

# Partial regularity for doubly nonlinear parabolic systems of the first type

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## Abstract

We study solutions  $\mathbf{v}$  of the parabolic system of PDE

$$\partial_t (D\psi(\mathbf{v})) = \operatorname{div} DF(D\mathbf{v}).$$

Here  $\psi$  and  $F$  are convex functions, and this is a model equation for more general doubly nonlinear evolutions that arise in the study of phase transitions in materials. We show that if  $\mathbf{v}$  is a weak solution, then  $D\mathbf{v}$  is locally Hölder continuous except for possibly on a lower dimensional subset of the domain of  $\mathbf{v}$ . Our proof is based on compactness properties of solutions, two integral identities and a fractional time derivative estimate for  $D\mathbf{v}$ .

## 1 Introduction

A doubly nonlinear evolution is a flow that may involve a time derivative of a nonlinear function of a quantity of interest. These types of flows arise in various physical models for phase transitions as detailed in the monograph [46]. Notable examples include the Stefan problem, which concerns a classical model of phase transition in solid-liquid systems [18, 19, 32, 38, 41]; the Hele-Shaw and Muskat problems, which involve the dynamics of two immiscible viscous fluids [8, 35, 36, 39]; the flow of an incompressible fluid through a porous medium [1, 13]; and the study of interfacial (Gibbs–Thomson) effects that occur during phase nucleation, growth and coarsening [7, 23, 24, 33, 47]. In this paper, we will consider doubly nonlinear evolutions that are also parabolic systems.

In what follows, we will focus on solutions  $\mathbf{v} : U \times (0, T) \rightarrow \mathbb{R}^m$  of PDE systems of the form

$$\partial_t (D\psi(\mathbf{v})) = \operatorname{div} DF(D\mathbf{v}). \tag{1.1}$$

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Here  $U \subset \mathbb{R}^n$  is a bounded domain with smooth boundary,  $T > 0$ , and  $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $F : \mathbb{M}^{m \times n} \rightarrow \mathbb{R}$  are convex. We will denote  $\mathbb{M}^{m \times n}$  as the space of  $m \times n$  matrices with real entries and  $\mathbf{v} = (v^1, \dots, v^m)$  for the  $m$  component functions  $v^i = v^i(x, t)$  of  $\mathbf{v}$ . Also note

$$\mathbf{v}_t = (v_t^1, \dots, v_t^m) \in \mathbb{R}^m \quad \text{and} \quad D\mathbf{v} = \begin{pmatrix} v_{x_1}^1 & \dots & v_{x_n}^1 \\ & \ddots & \\ v_{x_1}^m & \dots & v_{x_n}^m \end{pmatrix} \in \mathbb{M}^{m \times n}$$

are the respective time derivative and spatial gradient matrix of  $\mathbf{v}$ .

Writing  $\psi(w) = \psi(w^1, \dots, w^m)$  for  $w = (w^i) \in \mathbb{R}^m$  and  $F(M) = F(M_1^1, \dots, M_n^m)$  for  $M = (M_j^i) \in \mathbb{M}^{m \times n}$ , the system (1.1) can also be posed as the system of  $m$  equations

$$\partial_t (\psi_{w_i}(\mathbf{v})) = \sum_{j=1}^n \left( F_{M_j^i}(D\mathbf{v}) \right)_{x_j}, \quad i = 1, \dots, m.$$

Carrying out the derivatives, we may also write

$$\sum_{j=1}^n \psi_{w_i w_j}(\mathbf{v}) v_t^j = \sum_{k=1}^m \sum_{j,\ell=1}^n F_{M_j^i M_\ell^k}(D\mathbf{v}) v_{x_\ell x_j}^k, \quad i = 1, \dots, m.$$

In particular, we see that the system (1.1) is a type of quasilinear system of parabolic PDE.

The principle assumptions that we will make in this work are that there are constants  $\theta, \lambda, \Theta, \Lambda > 0$  such that

$$\theta |w_1 - w_2|^2 \leq (D\psi(w_1) - D\psi(w_2)) \cdot (w_1 - w_2) \leq \Theta |w_1 - w_2|^2 \quad (1.2)$$

for each  $w_1, w_2 \in \mathbb{R}^m$  and

$$\lambda |M_1 - M_2|^2 \leq (DF(M_1) - DF(M_2)) \cdot (M_1 - M_2) \leq \Lambda |M_1 - M_2|^2 \quad (1.3)$$

for each  $M_1, M_2 \in \mathbb{M}^{m \times n}$ . Here we are using the notation  $M \cdot N := \text{tr}(M^t N)$  and  $|M| := (M \cdot M)^{1/2}$  for each  $M, N \in \mathbb{M}^{m \times n}$ . In other words,  $\psi$  and  $F$  will always assumed to be (at least) continuously differentiable, uniformly convex and to grow quadratically. We also point out that these assumptions require  $D\psi$  and  $DF$  to be globally Lipschitz mappings which are therefore differentiable almost everywhere on  $\mathbb{R}^m$  and  $\mathbb{M}^{m \times n}$ , respectively.

One of the central results of this paper is as follows. We remark that we have postponed the definition of weak solution (Definition 3.1) until later in this work.

**Theorem 1.** *Assume  $\psi \in C^2(\mathbb{R}^m)$  and  $F \in C^2(\mathbb{M}^{m \times n})$  and that (1.2) and (1.3) hold. Suppose  $\mathbf{v}$  is a weak solution of (1.1). Then there is an open subset  $\mathcal{O} \subset U \times (0, T)$  whose complement has Lebesgue measure 0 for which  $D\mathbf{v}$  is Hölder continuous in a neighborhood of each point in  $\mathcal{O}$ .*

Our approach to proving Theorem 1 is as follows. First, we will derive two integral identities for weak solutions which provides us with some local energy estimates. Next, we will establish that every appropriately bounded sequence of solutions of the system (1.1) has a subsequence that converges strongly to another weak solution (1.1). Then we will exploit this compactness to argue that certain integral quantities decay as they would for solutions of the linearization of (1.1). Finally, we will use versions of Lebesgue's differentiation theorem and Campanato's criterion with parabolic cylinders (instead of Euclidean balls) to obtain partial Hölder continuity of the gradients of weak solutions.

It turns out that it is also possible to establish a certain fractional time differentiability of  $D\mathbf{v}$ . This property can be used to demonstrate that the set  $\mathcal{O}$  in the statement of Theorem 1 can be selected to be lower dimensional. We state the following refinement of Theorem 1 postponing the definition of Parabolic Hausdorff measure  $\mathcal{P}^s$  ( $0 \leq s \leq n+2$ ) until the final section of this paper (Definition 4.5). We only note here that the Lebesgue outer measure on  $\mathbb{R}^{n+1}$  is absolutely continuous with respect to  $\mathcal{P}^{n+2}$ , which means that the following assertion improves Theorem 1.

**Theorem 2.** *Assume  $\psi \in C^2(\mathbb{R}^m)$  and  $F \in C^2(\mathbb{M}^{m \times n})$  and that (1.2) and (1.3) hold. Suppose  $\mathbf{v}$  is a weak solution of (1.1) and*

$$\mathcal{O} = \{(x, t) \in U \times (0, T) : D\mathbf{v} \text{ is Hölder continuous in some neighborhood of } (x, t)\}.$$

*Then there is  $\beta \in (0, 1)$  for which*

$$\mathcal{P}^{n+2-2\beta}(U \times (0, T) \setminus \mathcal{O}) = 0.$$

We note that partial regularity statements such as what is claimed above are quite natural to consider for systems. Even in the stationary case of (1.1), weak solutions  $\mathbf{u} : U \rightarrow \mathbb{R}^m$  of

$$-\operatorname{div}(DF(D\mathbf{u})) = 0$$

may fail to be regular at some points [11, 22, 30]. We also note that Theorems 1 and 2 have already been established for the system

$$\partial_t \mathbf{v} = \operatorname{div} DF(D\mathbf{v}),$$

which corresponds to (1.1) when  $\psi(w) = \frac{1}{2}|w|^2$  [6]. In more recent work [14, 15], the stronger assertion  $\mathcal{P}^{n-\delta}(U \times (0, T) \setminus \mathcal{O}) = 0$  for some  $\delta > 0$  was verified. It is also worth mentioning that with appropriate growth and regularity assumptions on  $F$  and boundary conditions, this system can be interpreted as an  $L^2(U; \mathbb{R}^m)$  gradient flow related to  $\mathbf{u} \mapsto \int_U F(D\mathbf{u})dx$  [4, 9].

When  $F(M) = \frac{1}{2}|M|^2$ , (1.1) becomes  $\partial_t(D\psi(\mathbf{v})) = \Delta \mathbf{v}$ . Setting  $\mathbf{w} = D\psi(\mathbf{v})$  gives

$$\partial_t \mathbf{w} = \Delta(D\psi^*(\mathbf{w})), \tag{1.4}$$

where  $\psi^*$  is the Legendre transform of  $\psi$ . The system (1.4) is a generalization of the porous medium equation and other nonlinear diffusion equations. Moreover, if  $\psi$  is uniformly convex

and grows quadratically and  $\mathbf{w}|_{\partial U} = 0$ , (1.4) corresponds to an  $H^{-1}(U; \mathbb{R}^m) = (H_0^1(U; \mathbb{R}^m))^*$  gradient flow related to  $\mathbf{u} \mapsto \int_U \psi^*(\mathbf{u}) dx$  [43]. It has been established that weak solutions of (1.4) are Hölder continuous in an open set  $\mathcal{O}$  where  $\mathcal{P}^{n-2}(U \times (0, T) \setminus \mathcal{O}) = 0$  [21, 43]. Theorems 1 and 2 complement these results as they involve the Hölder continuity of the gradient of solutions of (1.4).

Observe that when  $m = 1$ , the system (1.1) reduces to a single PDE

$$\partial_t(\psi'(v)) = \operatorname{div}(DF(Dv))$$

for a scalar function  $v : U \times (0, T) \rightarrow \mathbb{R}$ . For many examples of  $\psi$  and  $F$  that satisfy suitable regularity, convexity and compatible growth conditions, there is a local Hölder estimate for solutions [28, 34, 45]. For example, Hölder regularity has been shown in various contexts for the particular case

$$\partial_t(|v|^{p-2}v) = \operatorname{div}(|Dv|^{p-2}Dv). \quad (1.5)$$

[29, 44]. It would be very interesting to pursue the regularity of systems with analogous growth and convexity properties to (1.5).

Finally, we remark that equation (1.1) is known as a doubly nonlinear parabolic system of the *first* type. A doubly nonlinear parabolic system of the *second* type is of the form

$$D\psi(\mathbf{v}_t) = \operatorname{div} DF(D\mathbf{v}). \quad (1.6)$$

We believe this terminology is due to A. Visintin [46]. The main difference between (1.1) and (1.6) is that (1.1) is quasilinear while (1.6) is fully nonlinear. Nevertheless, solutions of both systems exhibit partial regularity. In forthcoming work [25], we will prove analogs of Theorem 1 and 2 for solutions of (1.6).

## 2 Formal computations

Our first task will be to briefly derive two simple, yet important integral identities for solutions of (1.1). They each imply useful local energy estimates. We will establish these identities first for smooth solutions (1.1) and then in the following section we will prove them for weak solutions. In fact, the corresponding estimates dictate the spaces that one may expect weak solutions to belong to; so these estimates actually guide much of the analysis to follow.

**Proposition 2.1.** *Let  $\psi^*$  be the Legendre transform of  $\psi$ . Assume  $\mathbf{v} \in C^\infty(U \times (0, T); \mathbb{R}^m)$  is a solution of (1.1) and  $\phi \in C_c^\infty(U \times (0, \infty))$ . Then*

$$\frac{d}{dt} \int_U \psi^*(D\psi(\mathbf{v})) \phi dx + \int_U \phi DF(D\mathbf{v}) \cdot D\mathbf{v} dx = \int_U (\psi^*(D\psi(\mathbf{v})) \phi_t - \mathbf{v} \cdot DF(D\mathbf{v}) D\phi) dx. \quad (2.1)$$

*Proof.* By direct computation, we have

$$\frac{d}{dt} \int_U \psi^*(D\psi(\mathbf{v})) \phi dx = \frac{d}{dt} \int_U \phi (D\psi(\mathbf{v}) \cdot \mathbf{v} - \psi(\mathbf{v})) dx$$

$$\begin{aligned}
&= \int_U \phi_t (D\psi(\mathbf{v}) \cdot \mathbf{v} - \psi(\mathbf{v})) dx + \int_U \phi \partial_t (D\psi(\mathbf{v})) \cdot \mathbf{v} dx \\
&= \int_U \phi_t \psi^* (D\psi(\mathbf{v})) dx + \int_U \phi \operatorname{div} [DF(D\mathbf{v})] \cdot \mathbf{v} dx \\
&= \int_U \phi_t \psi^* (D\psi(\mathbf{v})) dx - \int_U \mathbf{v} \cdot DF(D\mathbf{v}) D\phi dx - \int_U \phi DF(D\mathbf{v}) \cdot D\mathbf{v} dx.
\end{aligned}$$

□

Now suppose  $O$  is the  $m \times n$  matrix of zeros. Notice that we have

$$\partial_t (D\psi(\mathbf{v}) - D\psi(0)) = \operatorname{div} [DF(D\mathbf{v}) - DF(O)].$$

Consequently, upon subtracting  $\psi(0) + D\psi(0) \cdot w$  from  $\psi(w)$  and subtracting  $F(O) + DF(O) \cdot M$  from  $F(M)$ , we may assume without any loss of generality that

$$\psi(0) = |D\psi(0)| = 0 \quad \text{and} \quad F(O) = |DF(O)| = 0. \quad (2.2)$$

With this assumption and (1.2),

$$\begin{cases} \frac{1}{2}\theta|w|^2 \leq \psi(w) \leq \frac{1}{2}\Theta|w|^2 \\ \frac{1}{2}\theta|w|^2 \leq D\psi(w) \cdot w - \psi(w) \leq \frac{1}{2}\Theta|w|^2 \\ \theta|w|^2 \leq D\psi(w) \cdot w \leq \Theta|w|^2 \\ \theta|w| \leq |D\psi(w)| \leq \Theta|w|. \end{cases} \quad (2.3)$$

Analogous inequalities hold for  $F$  and  $DF$ , as well.

**Corollary 2.2.** *Assume  $\mathbf{v} \in C^\infty(U \times (0, T); \mathbb{R}^m)$  is a solution of (1.1). There is a constant  $C = C(\lambda, \Lambda, \theta, \Theta)$  such that for each  $\eta \in C_c^\infty(U \times (0, \infty))$  with  $\eta \geq 0$ ,*

$$\max_{0 \leq t \leq T} \int_U \eta^2 |\mathbf{v}|^2 dx + \int_0^T \int_U \eta^2 |D\mathbf{v}|^2 dx dt \leq C \int_0^T \int_U (\eta |\eta_t| + |D\eta|^2) |\mathbf{v}|^2 dx dt. \quad (2.4)$$

*Proof.* Let  $\phi = \eta^2$  in (2.1). Employing (1.2) and (2.3) and integrating this identity from  $[0, t]$  gives

$$\begin{aligned}
&\frac{\theta}{2} \int_U \eta(x)^2 |\mathbf{v}(x, t)|^2 dx + \lambda \int_0^t \int_U \eta^2 |D\mathbf{v}|^2 dx ds \\
&\leq \int_U \eta(x)^2 \psi^* (D\psi(\mathbf{v}(x, t))) dx + \int_0^t \int_U \eta^2 D\mathbf{v} \cdot DF(D\mathbf{v}) dx ds \\
&= \int_0^t \int_U (\psi^* (D\psi(\mathbf{v})) 2\eta \eta_t - \mathbf{v} \cdot DF(D\mathbf{v}) (2\eta D\eta)) dx ds \\
&\leq \int_0^t \int_U (\Theta |\mathbf{v}|^2 \eta |\eta_t| + |\mathbf{v}| \cdot \Lambda |D\mathbf{v}| 2\eta |D\eta|) dx ds
\end{aligned}$$

$$\begin{aligned}
&= \int_0^t \int_U \left( \Theta |\mathbf{v}|^2 \eta |\eta_t| + \left( \frac{2\Lambda}{\sqrt{\lambda}} |D\eta| |\mathbf{v}| \right) \cdot (\sqrt{\lambda} \eta |D\mathbf{v}|) \right) dx ds \\
&\leq \left( \Theta + \frac{2\Lambda^2}{\lambda} \right) \int_0^t \int_U (\eta |\eta_t| + |D\eta|^2) |\mathbf{v}|^2 dx ds + \frac{\lambda}{2} \int_0^t \int_U \eta^2 |D\mathbf{v}|^2 dx ds.
\end{aligned}$$

Consequently, the assertion holds with

$$C = \frac{\Theta + \frac{2\Lambda^2}{\lambda}}{\frac{1}{2} \min\{\theta, \lambda\}}.$$

□

One may interpret identity (2.1) as the result of multiplying (1.1) by  $\phi \mathbf{v}$  and integrating by parts. We will now discuss another identity which can be obtained in a similar fashion by multiplying (1.1) by  $\phi \mathbf{v}_t$  and integrating by parts.

**Proposition 2.3.** *Assume  $\mathbf{v} \in C^\infty(U \times (0, T); \mathbb{R}^m)$  is a solution of (1.1) and  $\phi \in C_c^\infty(U \times (0, \infty))$ . Then*

$$\frac{d}{dt} \int_U \phi F(D\mathbf{v}) dx + \int_U \phi \partial_t (D\psi(\mathbf{v})) \cdot \mathbf{v}_t dx = \int_U (\phi_t F(D\mathbf{v}) - \mathbf{v}_t \cdot DF(D\mathbf{v}) D\phi) dx. \quad (2.5)$$

*Proof.* We compute

$$\begin{aligned}
\frac{d}{dt} \int_U \phi F(D\mathbf{v}) dx &= \int_U \phi_t F(D\mathbf{v}) dx + \int_U \phi DF(D\mathbf{v}) \cdot D\mathbf{v}_t dx \\
&= \int_U \phi_t F(D\mathbf{v}) dx - \int_U \mathbf{v}_t \cdot \operatorname{div}(\phi DF(D\mathbf{v})) dx \\
&= \int_U \phi_t F(D\mathbf{v}) dx - \int_U \mathbf{v}_t \cdot (DF(D\mathbf{v}) D\phi + \phi \operatorname{div}(DF(D\mathbf{v}))) dx \\
&= \int_U \phi_t F(D\mathbf{v}) dx - \int_U \mathbf{v}_t \cdot (DF(D\mathbf{v}) D\phi + \phi \partial_t (D\psi(\mathbf{v}))) dx.
\end{aligned}$$

□

**Corollary 2.4.** *Assume  $\mathbf{v} \in C^\infty(U \times (0, T); \mathbb{R}^m)$  is a solution of (1.1). There is a constant  $C = C(\lambda, \Lambda, \theta)$  such that for each  $\eta \in C_c^\infty(U \times (0, \infty))$  with  $\eta \geq 0$ ,*

$$\max_{0 \leq t \leq T} \int_U \eta^2 |D\mathbf{v}|^2 dx + \int_0^T \int_U \eta^2 |\mathbf{v}_t|^2 dx dt \leq C \int_0^T \int_U (\eta |\eta_t| + |D\eta|^2) |D\mathbf{v}|^2 dx dt. \quad (2.6)$$

*Proof.* We choose  $\phi = \eta^2$  and argue similar to how we did in deriving (2.4). In particular, using (1.2), (1.3) and (2.2) and integrating (2.5) from  $[0, t]$  leads to

$$\frac{\lambda}{2} \int_U \eta(x)^2 |D\mathbf{v}(x, t)|^2 dx + \theta \int_0^t \int_U \eta^2 |\mathbf{v}_t|^2 dx ds$$

$$\begin{aligned}
&\leq \int_U \eta(x)^2 F(D\mathbf{v}(x, t)) dx + \int_0^t \int_U \eta^2 D^2 \psi(\mathbf{v}) \mathbf{v}_t \cdot \mathbf{v}_t dx ds \\
&= \int_U \eta(x)^2 F(D\mathbf{v}(x, t)) dx + \int_0^t \int_U \eta^2 \partial_t (D\psi(\mathbf{v})) \cdot \mathbf{v}_t dx ds \\
&= \int_0^t \int_U (2\eta \eta_t F(D\mathbf{v}) - \mathbf{v}_t \cdot DF(D\mathbf{v}) 2\eta D\eta) dx ds \\
&\leq \int_0^t \int_U (\Lambda |D\mathbf{v}|^2 \eta |\eta_t| + \eta |\mathbf{v}_t| \cdot 2\Lambda |D\mathbf{v}| |D\eta|) dx ds \\
&= \Lambda \int_0^t \int_U \left( |D\mathbf{v}|^2 \eta |\eta_t| + \sqrt{\theta} \eta |\mathbf{v}_t| \cdot \frac{2}{\sqrt{\theta}} |D\mathbf{v}| |D\eta| \right) dx ds \\
&\leq \Lambda \left( 1 + \frac{2}{\theta} \right) \int_0^t \int_U (\eta |\eta_t| + |D\eta|^2) |D\mathbf{v}|^2 dx ds + \frac{\theta}{2} \int_0^t \int_U \eta^2 |\mathbf{v}_t|^2 dx ds.
\end{aligned}$$

Therefore, the conclusion holds with

$$C = \frac{\Lambda \left( 1 + \frac{2}{\theta} \right)}{\frac{1}{2} \min\{\theta, \lambda\}}.$$

□

### 3 Weak Solutions

In view of estimates (2.4) and (2.6) and the natural divergence structure of (1.1), we define weak solutions as follows.

**Definition 3.1.** A measurable mapping  $\mathbf{v} : U \times (0, T) \rightarrow \mathbb{R}^m$  is a *weak solution* of (1.1) in  $U \times (0, T)$  if  $\mathbf{v}$  satisfies

$$\mathbf{v} \in L_{\text{loc}}^\infty((0, T); H_{\text{loc}}^1(U; \mathbb{R}^m)) \quad \text{and} \quad \mathbf{v}_t \in L_{\text{loc}}^2(U \times (0, T); \mathbb{R}^m) \quad (3.1)$$

and

$$\int_0^T \int_U D\psi(\mathbf{v}) \cdot \mathbf{w}_t dx dt = \int_0^T \int_U DF(D\mathbf{v}) \cdot D\mathbf{w} dx dt, \quad (3.2)$$

for all  $\mathbf{w} \in C_c^\infty(U \times (0, T); \mathbb{R}^m)$ .

In this section, we will make some crucial observations about the integrability of weak solutions and show that weak solutions satisfy the identities (2.1) and (2.5). Then we will establish that weak solutions have a compactness property, which will be vital to our proof of Theorem 1. We also note that in Appendix A, we show how to construct a weak solution of the initial value problem associated with (1.1) where solutions satisfy the Dirichlet boundary condition.

### 3.1 Estimates

We will now proceed to derive some continuity and estimates of weak solutions. As usual, we will denote  $H^{-1}(V; \mathbb{R}^m)$  for the continuous dual space to  $H_0^1(V; \mathbb{R}^m)$  for each open  $V \subset U$ .

**Lemma 3.2.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) on  $U \times (0, T)$ ,  $[t_0, t_1] \subset (0, T)$  and  $V \subset\subset U$  is open. Then*

$$\mathbf{v} : [t_0, t_1] \rightarrow H^1(V; \mathbb{R}^m) \text{ is weakly continuous} \quad (3.3)$$

and

$$D\psi \circ \mathbf{v} : [t_0, t_1] \rightarrow H^{-1}(V; \mathbb{R}^m) \text{ is Lipschitz continuous.} \quad (3.4)$$

*Proof.* In view of (3.1),  $\mathbf{v}_t \in L^2([t_0, t_1]; L^2(V; \mathbb{R}^m))$ . Therefore,  $\mathbf{v} : [t_0, t_1] \rightarrow L^2(V; \mathbb{R}^m)$  is absolutely continuous. Now let  $t_k \in [t_0, t_1]$  with  $t_k \rightarrow t$ ; we clearly have  $\mathbf{v}(\cdot, t_k) \rightarrow \mathbf{v}(\cdot, t)$  in  $L^2(V; \mathbb{R}^m)$ . We also have by (3.1) that  $(D\mathbf{v}(\cdot, t_k))_{k \in \mathbb{N}} \subset L^2(V; \mathbb{M}^{m \times n})$  is bounded and hence has a weakly convergent subsequence  $(D\mathbf{v}(\cdot, t_{k_j}))_{j \in \mathbb{N}}$ . It is routine to check that the weak limit of this subsequence must be  $D\mathbf{v}(\cdot, t)$ . Since this limit is independent of the subsequence, it follows that  $D\mathbf{v}(\cdot, t_k) \rightharpoonup D\mathbf{v}(\cdot, t)$  in  $L^2(V; \mathbb{M}^{m \times n})$ . We conclude (3.3).

The weak solution condition (3.2) implies

$$\frac{d}{dt} \int_U D\psi(\mathbf{v}(x, t)) \cdot \mathbf{u}(x) dx + \int_U DF(D\mathbf{v}(x, t)) \cdot D\mathbf{u}(x) dx = 0 \quad (3.5)$$

in the sense of distributions on  $(0, T)$  for each  $\mathbf{u} \in H_0^1(V; \mathbb{R}^m) \subset H_0^1(U; \mathbb{R}^m)$ . In particular, (3.5) holds for almost every  $t \in (0, T)$  (Chapter 3, Lemma 1.1 of [42]). It follows that  $D\psi(\mathbf{v}) : [t_0, t_1] \rightarrow H^{-1}(V; \mathbb{R}^m)$  is differentiable almost everywhere on  $[t_0, t_1]$  and that

$$\|\partial_t (D\psi(\mathbf{v}(\cdot, t)))\|_{H^{-1}(V; \mathbb{R}^m)} = \|DF(D\mathbf{v}(\cdot, t))\|_{L^2(V; \mathbb{M}^{m \times n})} \leq \Lambda \|D\mathbf{v}(\cdot, t)\|_{L^2(V; \mathbb{M}^{m \times n})}$$

for almost every  $t \in [t_0, t_1]$ . In view of (3.1), we deduce (3.4).  $\square$

We can also use elliptic regularity results to conclude some integrability of the Hessian of  $\mathbf{v} = (v^1, \dots, v^m)$ . Below we use the notation

$$D^2\mathbf{v} := (D^2v^1, \dots, D^2v^m) \in (\mathbb{M}^{n \times n})^m$$

and

$$|D^2\mathbf{v}|^2 := \sum_{i=1}^m |D^2v^i|^2 = \sum_{i=1}^m \sum_{j,k=1}^n (v_{x_j x_k}^i)^2.$$

**Lemma 3.3.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) on  $U \times (0, T)$ . Then*

$$D^2\mathbf{v} \in L_{loc}^2(U \times (0, T); (\mathbb{M}^{n \times n})^m). \quad (3.6)$$

*In particular, equation (1.1) holds almost everywhere in  $U \times (0, T)$ .*



*Proof.* In view of (3.1) and the Lipschitz continuity of  $D\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,

$$\partial_t(D\psi \circ \mathbf{v})(\cdot, t) = D^2\psi(\mathbf{v}(\cdot, t))\mathbf{v}_t(\cdot, t)$$

in  $L^2_{\text{loc}}(U; \mathbb{R}^m)$  for almost every  $t \in (0, T)$ . In view of (3.2),

$$D^2\psi(\mathbf{v}(\cdot, t))\mathbf{v}_t(\cdot, t) = \text{div} DF(D\mathbf{v}(\cdot, t)) \quad (3.7)$$

weakly in  $U$  for almost every  $t \in (0, T)$ .

Let  $W, V \subset U$  be open with  $W \subset\subset V \subset\subset U$ . As  $\mathbf{v}(\cdot, t)$  satisfies (3.7), the associated  $W^{2,2}_{\text{loc}}(U)$  estimates (Proposition 8.6 in [20] or Theorem 1, Section 8.3 of [17]) for uniformly elliptic Euler-Lagrange equations imply  $D^2\mathbf{v}(\cdot, t) \in L^2_{\text{loc}}(U; (\mathbb{M}^{n \times n})^m)$  and

$$\begin{aligned} \int_W |D^2\mathbf{v}(x, t)|^2 dx &\leq C \int_V (|D^2\psi(\mathbf{v}(x, t))\mathbf{v}_t(x, t)|^2 + |D\mathbf{v}(x, t)|^2) dx \\ &\leq C \int_V (\Lambda^2 |\mathbf{v}_t(x, t)|^2 + |D\mathbf{v}(x, t)|^2) dx \end{aligned}$$

for almost every  $t \in (0, T)$ . Here  $C$  is a constant that is independent of  $\mathbf{v}$ . The assertion (3.6) now follows from integrating the above inequality locally in time.

Now that we have also established (3.6), we can integrate by parts in (3.2) to get

$$\int_0^T \int_U [\partial_t(D\psi(\mathbf{v})) - \text{div}(DF(D\mathbf{v}))] \cdot \mathbf{w} dx dt = 0,$$

for all  $\mathbf{w} \in C_c^\infty(U \times (0, T); \mathbb{R}^m)$ . Thus  $\partial_t(D\psi(\mathbf{v})) = \text{div}(DF(D\mathbf{v}))$  almost everywhere in  $U \times (0, T)$ .  $\square$

The conclusion of the previous lemma sets up an application of the interpolation of Lebesgue spaces and the Gagliardo-Nirenberg-Sobolev inequality to improve the space time integrability of the gradient of weak solutions. We refer also to Lemma 5.3 in [15] for a more general result.

**Corollary 3.4.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) on  $U \times (0, T)$ . There exists an exponent  $p > 2$  such that*

$$D\mathbf{v} \in L^p_{\text{loc}}(U \times (0, T); \mathbb{M}^{m \times n}). \quad (3.8)$$

*Proof.* Let  $n \geq 3$ ,  $\eta \in C_c^\infty(U)$  and  $r \in (2, 2^*)$ . Here we are using the notation

$$2^* := \frac{2n}{n-2}.$$

Also select  $\lambda \in (0, 1)$  so that  $r = \lambda \cdot 2 + (1 - \lambda) \cdot 2^*$ ; that is  $\lambda = (2^* - r)/(2^* - 2)$ . By the interpolation of the Lebesgue spaces and the Gagliardo-Nirenberg-Sobolev inequality, there is a constant  $C_0$  depending only on  $r$  and  $n$  such that

$$\int_U |\eta D\mathbf{v}|^r dx \leq \left( \int_U |\eta D\mathbf{v}|^2 dx \right)^\lambda \left( \int_U |\eta D\mathbf{v}|^{2^*} dx \right)^{1-\lambda}$$

$$\begin{aligned}
&\leq C_0 \left( \int_U |\eta D\mathbf{v}|^2 dx \right)^\lambda \left( \int_U |D(\eta D\mathbf{v})|^2 dx \right)^{\frac{2^*}{2}(1-\lambda)} \\
&\leq 2^{\frac{2^*}{2}(1-\lambda)} C_0 \left( \int_U |\eta D\mathbf{v}|^2 dx \right)^\lambda \left( \int_U (\eta^2 |D^2 \mathbf{v}|^2 + |D\mathbf{v}|^2 |D\eta|^2) dx \right)^{\frac{2^*}{2}(1-\lambda)}.
\end{aligned}$$

Here we have suppressed the time dependence of  $D\mathbf{v}$  and  $D^2 \mathbf{v}$ ; however, we note that the above inequality holds for almost every  $t \in (0, T)$ .

Now we select  $r$  such that

$$\frac{2^*}{2}(1-\lambda) = \frac{2^*}{2} \frac{r-2}{2^*-2} = 1.$$

Namely, we choose  $r = 2 + \frac{4}{n} \in (2, 2^*)$  to get

$$\int_U |\eta D\mathbf{v}|^{2+\frac{4}{n}} dx \leq 2C \left( \int_U |\eta D\mathbf{v}|^2 dx \right)^\lambda \left( \int_U (\eta^2 |D^2 \mathbf{v}|^2 + |D\mathbf{v}|^2 |D\eta|^2) dx \right).$$

The assertion then follows for  $n \geq 3$  by recalling  $D\mathbf{v} \in L_{\text{loc}}^\infty((0, T); L_{\text{loc}}^2(U; \mathbb{M}^{m \times n}))$ , invoking Lemma 3.3 and integrating locally in time. For  $n = 2$ , a similar computation can be made to show that  $D\mathbf{v} \in L_{\text{loc}}^p(U \times (0, T); \mathbb{M}^{m \times 2})$  for each  $p \in [2, 4)$ .

Now let  $n = 1$ ,  $V \subset\subset U$  be an interval and  $[t_0, t_1] \in (0, T)$ . By Lemma 3.3,

$$\mathbf{v}_x(x, t) = \mathbf{v}_x(y, t) + \int_y^x \mathbf{v}_{xx}(z, t) dz$$

for  $x, y \in V$  and almost every  $t \in [t_0, t_1]$ . It then follows that

$$|\mathbf{v}_x(x, t)|^2 \leq 2 \left( |\mathbf{v}_x(y, t)|^2 + |V| \int_V |\mathbf{v}_{xx}(z, t)|^2 dz \right).$$

Integrating over  $y \in V$  gives

$$|V| |\mathbf{v}_x(x, t)|^2 \leq 2 \left( \int_V |\mathbf{v}_x(y, t)|^2 dy + |V|^2 \int_V |\mathbf{v}_{xx}(z, t)|^2 dz \right).$$

Consequently,  $\mathbf{v}_x \in L^2([t_0, t_1]; L^\infty(V; \mathbb{R}^m))$ . Combining with  $\mathbf{v}_x \in L^\infty([t_0, t_1]; L^2(V; \mathbb{R}^m))$  leads to

$$\begin{aligned}
\int_{t_0}^{t_1} \int_V |\mathbf{v}_x(x, t)|^4 dx dt &= \int_{t_0}^{t_1} \int_V |\mathbf{v}_x(x, t)|^2 |\mathbf{v}_x(x, t)|^2 dx dt \\
&\leq \int_{t_0}^{t_1} \|\mathbf{v}_x(\cdot, t)\|_{L^\infty(V; \mathbb{R}^m)}^2 \left( \int_V |\mathbf{v}_x(x, t)|^2 dx \right) dt \\
&\leq \left( \int_{t_0}^{t_1} \|\mathbf{v}_x(\cdot, t)\|_{L^\infty(V; \mathbb{R}^m)}^2 dt \right) \left( \text{ess sup}_{t \in [t_0, t_1]} \int_V |\mathbf{v}_x(x, t)|^2 dx \right) \\
&< \infty.
\end{aligned}$$

Thus  $\mathbf{v}_x \in L_{\text{loc}}^4(U \times (0, T); \mathbb{R}^m)$ . □

### 3.2 Integral identities

We now show that identities (2.1) and (2.5) hold for weak solutions of (1.1).

**Proposition 3.5.** *Suppose  $\phi \in C_c^\infty(U \times (0, T))$  and  $\mathbf{v}$  is a weak solution of (1.1) on  $U \times (0, T)$ .  
(i)*

$$[0, T] \ni t \mapsto \int_U \psi^*(D\psi(\mathbf{v}(x, t))) \phi(x, t) dx$$

*is absolutely continuous and (2.1) holds for almost every  $t \in [0, T]$ .*

*(ii)*

$$[0, T] \ni t \mapsto \int_U F(D\mathbf{v}(x, t)) \phi(x, t) dx$$

*is absolutely continuous and (2.5) holds for almost every  $t \in [0, T]$ .*

*Proof.* 1. Suppose that the  $\phi$  is supported in  $V \times (t_0, t_1) \subset\subset U \times (0, T)$  for some open set  $V \subset \mathbb{R}^m$ . We have already remarked in our proof of Lemma 3.2 that  $\mathbf{v} : [t_0, t_1] \rightarrow L^2(V; \mathbb{R}^m)$  is absolutely continuous. Since  $D\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is Lipschitz and  $\psi^*(D\psi(w)) = D\psi(w) \cdot w - \psi(w)$ , it is routine to check that  $[0, T] \ni t \mapsto \int_U \psi^*(D\psi(\mathbf{v}(x, t))) \phi(x, t) dx$  is an absolutely continuous function. By Lemma 3.3, we also have that equation (1.1) holds almost everywhere in  $U \times (0, T)$  for weak solutions. Using virtually the same formal computations we made when deriving (2.1), we can show that (2.1) holds almost everywhere in  $(0, T)$ . We conclude assertion (i).

2. So we are left to prove assertion (ii). To this end, we let  $u \in C_c^\infty(U)$  be a function that is supported in  $V$  and suppose initially that  $u \geq 0$ . For a given  $\mathbf{w} \in L^2(V; \mathbb{R}^m)$ , we also define

$$\Phi(\mathbf{w}) := \begin{cases} \int_V u(x) F(D\mathbf{w}(x)) dx, & \mathbf{w} \in H^1(V; \mathbb{R}^m) \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that  $\Phi$  is proper, convex and lower-semicontinuous on  $L^2(V; \mathbb{R}^m)$ . Moreover, a routine computation shows

$$\begin{aligned} \partial\Phi(\mathbf{v}(\cdot, t)) &:= \left\{ \xi \in L^2(V; \mathbb{R}^m) : \Phi(\mathbf{u}) \geq \Phi(\mathbf{v}(\cdot, t)) + \int_V \xi \cdot (\mathbf{u} - \mathbf{v}(\cdot, t)) dx \text{ for all } \mathbf{u} \in L^2(V; \mathbb{R}^m) \right\} \\ &= \{-\operatorname{div}(u D F(D\mathbf{v}(\cdot, t)))\} \\ &= \{-D F(D\mathbf{v}(\cdot, t)) Du - u \partial_t(D\psi(\mathbf{v}(\cdot, t)))\} \end{aligned}$$

for almost every  $t \in (0, T)$ . Since  $-\operatorname{div}(u D F(D\mathbf{v})) \in L^2([t_0, t_1]; L^2(V; \mathbb{R}^m))$  and  $\mathbf{v}_t \in L^2([t_0, t_1]; L^2(V; \mathbb{R}^m))$ , it must be that  $\Phi \circ \mathbf{v}$  is locally absolutely continuous on  $(0, T)$  (Proposition 1.4.4 and Remark 1.4.6 [2]).

By the chain rule (Remark 1.4.6 [2]), we have by (3.5)

$$\frac{d}{dt} \int_U F(D\mathbf{v}(x, t)) u(x) dx = \frac{d}{dt} \int_V F(D\mathbf{v}(x, t)) u(x) dx$$

$$\begin{aligned}
&= \frac{d}{dt}(\Phi \circ \mathbf{v})(\cdot, t) \\
&= \int_V [-DF(D\mathbf{v}(\cdot, t))Du - u\partial_t(D\psi(\mathbf{v}(\cdot, t)))] \cdot \mathbf{v}_t dx \\
&= \int_U [-DF(D\mathbf{v}(\cdot, t))Du - u\partial_t(D\psi(\mathbf{v}(\cdot, t)))] \cdot \mathbf{v}_t dx \quad (3.9)
\end{aligned}$$

for almost every  $t \in (0, T)$ . In particular,

$$\begin{aligned}
\int_U F(D\mathbf{v}(x, t))u(x)dx &= \int_U F(D\mathbf{v}(x, s))u(x)dx + \\
&\quad \int_s^t \int_U [-DF(D\mathbf{v}(\cdot, \tau))Du - u\partial_\tau(D\psi(\mathbf{v}(\cdot, \tau)))] \cdot \mathbf{v}_\tau dx d\tau
\end{aligned} \quad (3.10)$$

for all  $t, s \in [0, T]$ .

3. For  $u \in C_c^\infty(U)$  that is not necessarily nonnegative, we may write  $u = u^+ - u^-$ . Let  $(u^\pm)^\epsilon := \eta^\epsilon * u^\pm$  be the standard mollification of  $u^\pm$ . Recall that  $\eta \in C_c^\infty(B_1(0))$  is a nonnegative, radial function that satisfies  $\int_{B_1(0)} \eta(z)dz = 1$  and  $\eta^\epsilon := \epsilon^{-n}\eta(\cdot/\epsilon)$ . Moreover, for all  $\epsilon > 0$  sufficiently small,  $(u^\pm)^\epsilon \in C^\infty(U)$ . Thus (3.10) holds with  $(u^+)^\epsilon$  and  $(u^-)^\epsilon$ . Subtracting the resulting equalities actually gives

$$\begin{aligned}
\int_U F(D\mathbf{v}(x, t))u^\epsilon(x)dx &= \int_U F(D\mathbf{v}(x, s))u^\epsilon(x)dx + \\
&\quad \int_s^t \int_U [-DF(D\mathbf{v}(\cdot, \tau))Du^\epsilon - u^\epsilon\partial_\tau(D\psi(\mathbf{v}(\cdot, \tau)))] \cdot \mathbf{v}_\tau dx d\tau
\end{aligned}$$

for  $t, s \in [0, T]$ . Therefore, sending  $\epsilon \rightarrow 0^+$  allows us to recover (3.10) for  $u$  without making any restrictions on the sign of  $u$ . In particular, (3.9) holds for any  $u \in C_c^\infty(U)$ .

4. Now let  $\eta \in C_c^\infty(U)$  be nonnegative and  $\eta|_V \equiv 1$ . We will use this function to show that  $\mathbf{v} : (0, T) \rightarrow H^1(V; \mathbb{R}^m)$  is continuous. Recall that we have already shown that this mapping is weakly continuous in (3.3). Moreover, (3.9) implies that  $(0, T) \ni t \mapsto \int_U \eta(x)F(D\mathbf{v}(x, t))dx$  is continuous. Now let  $t_k \in (0, T)$  with  $t_k \rightarrow t \in (0, T)$ . The uniform convexity of  $F$  gives

$$\begin{aligned}
\int_U F(D\mathbf{v}(x, t_k))\eta(x)dx &\geq \int_U F(D\mathbf{v}(x, t))\eta(x)dx \\
&\quad + \int_U DF(D\mathbf{v}(x, t)) \cdot (D\mathbf{v}(x, t_k) - D\mathbf{v}(x, t))\eta(x)dx \\
&\quad + \frac{\lambda}{2} \int_U |D\mathbf{v}(x, t_k) - D\mathbf{v}(x, t)|^2 \eta(x)dx \\
&\geq \int_U F(D\mathbf{v}(x, t))\eta(x)dx \\
&\quad + \int_U DF(D\mathbf{v}(x, t)) \cdot (D\mathbf{v}(x, t_k) - D\mathbf{v}(x, t))\eta(x)dx
\end{aligned}$$

$$+ \frac{\lambda}{2} \int_V |D\mathbf{v}(x, t_k) - D\mathbf{v}(x, t)|^2 dx.$$

Sending  $k \rightarrow \infty$ ,

$$\limsup_{k \rightarrow \infty} \int_V |D\mathbf{v}(x, t_k) - D\mathbf{v}(x, t)|^2 dx = 0.$$

This shows  $\mathbf{v} : (0, T) \rightarrow H^1(V; \mathbb{R}^m)$  is continuous.

5. Set  $f(t) := \int_U F(D\mathbf{v}(x, t))\phi(x, t)dx$  and let  $h \neq 0$ . Observe

$$\begin{aligned} \frac{f(t+h) - f(t)}{h} &= \int_U \frac{F(D\mathbf{v}(x, t+h))\phi(x, t+h) - F(D\mathbf{v}(x, t))\phi(x, t)}{h} dx \\ &= \int_U \phi(x, t) \left[ \frac{F(D\mathbf{v}(x, t+h)) - F(D\mathbf{v}(x, t))}{h} \right] dx \\ &\quad + \int_U F(D\mathbf{v}(x, t+h)) \left[ \frac{\phi(x, t+h) - \phi(x, t)}{h} \right] dx. \end{aligned}$$

By parts 2 and 3 of this proof,

$$\begin{aligned} \lim_{h \rightarrow 0} \int_U \phi(x, t) \left[ \frac{F(D\mathbf{v}(x, t+h)) - F(D\mathbf{v}(x, t))}{h} \right] dx = \\ - \int_U \mathbf{v}_t(x, t) \cdot DF(D\mathbf{v}(x, t))D\phi(x, t)dx - \int_U \phi(x, t)\partial_t(D\psi(\mathbf{v}(x, t))) \cdot \mathbf{v}_t(x, t)dx \end{aligned}$$

for almost every  $t \in (0, T)$ . From part 4 of this proof,

$$\lim_{h \rightarrow 0} \int_U F(D\mathbf{v}(x, t+h)) \left[ \frac{\phi(x, t+h) - \phi(x, t)}{h} \right] dx = \int_U F(D\mathbf{v}(x, t))\phi_t(x, t)dx$$

for every  $t \in (0, T)$ . Combining these limits completes a proof of (2.5). Finally, we note that if (2.5) holds then  $f$  is absolutely continuous as each term in (2.1) aside from the time derivative belongs to  $L^1[0, T]$ .  $\square$

**Corollary 3.6.** *Every weak solution of (1.1) in  $U \times (0, T)$  satisfies inequalities (2.4) and (2.6).*

### 3.3 Fractional time differentiability

In part 4 of the proof of Proposition 3.5, we showed that for each weak solution  $\mathbf{v}$  of (1.1) in  $U \times (0, T)$ ,  $\mathbf{v} : (0, T) \rightarrow H^1(V; \mathbb{R}^m)$  is continuous for every  $V \subset\subset U$ . This strengthened our previous assertion (3.3). Now we will build on these observations and establish a type of fractional time differentiability of  $D\mathbf{v}$ . The following estimate will be crucial to our proof of Theorem 2.

**Proposition 3.7.** Assume  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ , and let  $p > 2$  be the exponent in (3.8). For each open  $V \subset\subset U$  and  $[t_0, t_1] \in (0, T)$ , there is a constant  $C$  such that

$$\int_{t_0}^{t_1} \int_V |D\mathbf{v}(x, t+h) - D\mathbf{v}(x, t)|^2 dx dt \leq C|h|^{\frac{1}{2}-\frac{1}{p}} \quad (3.11)$$

for  $0 < |h| < \min\{1, t_0, T - t_1\}$ .

*Proof.* Let  $u \in C_c^\infty(U)$  be a nonnegative function satisfying  $u \equiv 1$  on  $V$ . We have from (3.9) that

$$\int_U u(x) F(D\mathbf{v}(x, t+h)) dx \leq \int_U u(x) F(D\mathbf{v}(x, t)) dx - \int_t^{t+h} \int_U \mathbf{v}_t \cdot DF(D\mathbf{v}) Du dx ds$$

for  $t \in [t_0, t_1]$  and  $h \in (0, \min\{1, t_0, T - t_1\})$ . By the uniform convexity of  $F$

$$\begin{aligned} \int_t^{t+h} \int_U \mathbf{v}_t \cdot DF(D\mathbf{v}) Du dx ds &\geq \int_U u(x) (F(D\mathbf{v}(x, t+h)) - F(D\mathbf{v}(x, t))) dx \\ &\geq \int_U u(x) DF(D\mathbf{v}(x, t)) \cdot (D\mathbf{v}(x, t+h) - D\mathbf{v}(x, t)) dx \\ &\quad + \frac{\lambda}{2} \int_U u(x) |D\mathbf{v}(x, t+h) - D\mathbf{v}(x, t)|^2 dx \\ &\geq - \int_V \operatorname{div}(u(x) DF(D\mathbf{v}(x, t))) \cdot (\mathbf{v}(x, t+h) - \mathbf{v}(x, t)) dx \\ &\quad + \frac{\lambda}{2} \int_V |D\mathbf{v}(x, t+h) - D\mathbf{v}(x, t)|^2 dx. \end{aligned} \quad (3.12)$$

By (3.1) and (3.8),  $|\mathbf{v}_t| |D\mathbf{v}| \in L_{\text{loc}}^{\frac{2p}{p+2}}(U \times (0, T); \mathbb{R}^m)$ . Therefore,

$$\begin{aligned} \int_t^{t+h} \int_U \mathbf{v}_t \cdot DF(D\mathbf{v}) Du dx ds &\leq C_0 \int_t^{t+h} \int_V |\mathbf{v}_t| |D\mathbf{v}| dx ds \\ &\leq C_0 \left( \int_{t_0}^{t_1} \int_V (|\mathbf{v}_t| |D\mathbf{v}|)^{\frac{2p}{p+2}} dx ds \right)^{\frac{1}{2} + \frac{1}{p}} (|V|h)^{\frac{1}{2} - \frac{1}{p}} \\ &\leq C_0 |V|^{\frac{1}{2} - \frac{1}{p}} \left( \int_{t_0}^{t_1} \int_V (|\mathbf{v}_t| |D\mathbf{v}|)^{\frac{2p}{p+2}} dx ds \right)^{\frac{1}{2} + \frac{1}{p}} h^{\frac{1}{2} - \frac{1}{p}}. \end{aligned} \quad (3.13)$$

Here  $C_0$  depends on  $\Lambda$  and  $\|Du\|_{L^\infty(U; \mathbb{R}^n)}$ . We can also use (3.1) to conclude

$$\begin{aligned} \int_V \operatorname{div}(u(x) DF(D\mathbf{v}(x, t))) \cdot (\mathbf{v}(x, t+h) - \mathbf{v}(x, t)) dx \\ \leq \left( \int_V |\operatorname{div}(u DF(D\mathbf{v}))|^2 dx \right)^{1/2} \left( \int_V |\mathbf{v}(x, t+h) - \mathbf{v}(x, t)|^2 dx \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_V |Du \cdot DF(D\mathbf{v}) + u \partial_t(D\psi(\mathbf{v}))|^2 dx \right)^{1/2} \left( h \int_t^{t+h} \int_V |\mathbf{v}_t|^2 dx ds \right)^{1/2} \\
&\leq C_1 \left( \int_V (|D\mathbf{v}|^2 + |\mathbf{v}_t|^2) dx \right)^{1/2} \left( \int_{t_0}^{t_1} \int_V |\mathbf{v}_t|^2 dx ds \right)^{1/2} h^{1/2}, \tag{3.14}
\end{aligned}$$

for a constant  $C_1$  depending on  $\Lambda, \Theta$  and  $\|u\|_{L^\infty(U)} + \|Du\|_{L^\infty(U; \mathbb{R}^n)}$ .

Combining (3.12), (3.13) and (3.14) gives

$$\begin{aligned}
&\frac{\lambda}{2} \int_{t_0}^{t_1} \int_V |D\mathbf{v}(x, t+h) - D\mathbf{v}(x, t)|^2 dx dt \\
&\leq C_0(t_1 - t_0) |V|^{\frac{1}{2} - \frac{1}{p}} \left( \int_{t_0}^{t_1} \int_V (|\mathbf{v}_t| |D\mathbf{v}|)^{\frac{2p}{p+2}} dx ds \right)^{\frac{1}{2} + \frac{1}{p}} h^{\frac{1}{2} - \frac{1}{p}} \\
&+ C_1 \left\{ \int_{t_0}^{t_1} \left( \int_V (|D\mathbf{v}|^2 + |\mathbf{v}_t|^2) dx \right)^{1/2} dt \right\} \left( \int_{t_0}^{t_1} \int_V |\mathbf{v}_t|^2 dx ds \right)^{1/2} h^{1/2} \\
&\leq C_0(t_1 - t_0) |V|^{\frac{1}{2} - \frac{1}{p}} \left( \int_{t_0}^{t_1} \int_V (|\mathbf{v}_t| |D\mathbf{v}|)^{\frac{2p}{p+2}} dx ds \right)^{\frac{1}{2} + \frac{1}{p}} h^{\frac{1}{2} - \frac{1}{p}} \\
&+ C_1 |t_1 - t_0|^{1/2} \left( \int_{t_0}^{t_1} \int_V (|D\mathbf{v}|^2 + |\mathbf{v}_t|^2) dx dt \right)^{1/2} \left( \int_{t_0}^{t_1} \int_V |\mathbf{v}_t|^2 dx ds \right)^{1/2} h^{1/2} \\
&\leq \left\{ C_0(t_1 - t_0) |V|^{\frac{1}{2} - \frac{1}{p}} \left( \int_{t_0}^{t_1} \int_V (|\mathbf{v}_t| |D\mathbf{v}|)^{\frac{2p}{p+2}} dx ds \right)^{\frac{1}{2} + \frac{1}{p}} \right. \\
&\quad \left. + C_1 |t_1 - t_0|^{1/2} \left( \int_{t_0}^{t_1} \int_V (|D\mathbf{v}|^2 + |\mathbf{v}_t|^2) dx dt \right)^{1/2} \left( \int_{t_0}^{t_1} \int_V |\mathbf{v}_t|^2 dx ds \right)^{1/2} \right\} h^{\frac{1}{2} - \frac{1}{p}}.
\end{aligned}$$

This computation establishes (3.11) for positive  $h$ . A similar argument establishes (3.11) for negative  $h$ . We leave the details to the reader.  $\square$

It now follows fairly routinely that  $D\mathbf{v}$  is fractionally differentiable with respect to time as exhibited in (3.16) below. The following assertion can be found in Proposition 3.4 [14] or Proposition 2.19 of [15].

**Corollary 3.8.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$  and  $p > 2$  is the exponent in (3.8). For each open  $V \subset\subset U$ ,  $[t_0, t_1] \in (0, T)$ , and*

$$\beta \in \left( 0, \frac{1}{2} - \frac{1}{p} \right), \tag{3.15}$$

*there is constant  $A = A(p, \beta, n, t_0, t_1, V) > 0$  such that*

$$\int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_V \frac{|D\mathbf{v}(x, t) - D\mathbf{v}(x, s)|^2}{|t - s|^{1+\beta}} dx dt ds \leq A \left( C + \|D\mathbf{v}\|_{L^2(V \times [t_0, t_1])}^2 \right). \tag{3.16}$$

*Here  $C$  is the constant in (3.11).*

### 3.4 Compactness

Now we will discuss the compactness properties of weak solutions. Roughly speaking, we will show that any “bounded” sequence of weak solutions to systems of the form (1.1) has a subsequence that converges “strongly” to another weak solution of this type of system. Our main tools will be the identities (2.1) and (2.5), the energy estimates (2.4) and (2.6) and a compactness result due to J. P. Aubin [3].

**Proposition 3.9.** *Let  $\psi^k \in C^1(\mathbb{R}^m)$  satisfy (1.2) and  $F^k \in C^1(\mathbb{M}^{m \times n})$  satisfy (1.3) for each  $k \in \mathbb{N}$ . Suppose  $(\mathbf{v}^k)_{k \in \mathbb{N}}$  is a sequence of weak solutions of*

$$\partial_t (D\psi^k(\mathbf{v}^k)) = \operatorname{div} DF^k(D\mathbf{v}^k)$$

*in  $U \times (0, T)$  and assume*

$$\sup_{k \in \mathbb{N}} \int_0^T \int_U (|\mathbf{v}^k|^2 + |D\mathbf{v}^k|^2) dx dt < \infty. \quad (3.17)$$

*Then there is a subsequence  $(\mathbf{v}^{k_j})_{j \in \mathbb{N}}$  and  $\mathbf{v} \in H^1(U \times (0, T); \mathbb{R}^m)$  such that for each open  $V \subset\subset U$  and interval  $[t_0, t_1] \subset (0, T)$ ,*

$$\mathbf{v}^{k_j} \rightarrow \mathbf{v} \text{ in } C([t_0, t_1]; H^1(V; \mathbb{R}^m)) \quad (3.18)$$

*and*

$$\mathbf{v}_t^{k_j} \rightarrow \mathbf{v}_t \text{ in } L^2(U \times (0, T); \mathbb{R}^m). \quad (3.19)$$

*Moreover, there is  $\psi \in C^1(\mathbb{R}^m)$  that satisfies (1.2) and  $F \in C^1(\mathbb{M}^{m \times n})$  that satisfies (1.3) for which  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ .*

*Proof.* 1. By hypothesis, we have that

$$|D\psi^k(z_1) - D\psi^k(z_2)| \leq \Theta |z_1 - z_2| \quad (z_1, z_2 \in \mathbb{R}^m)$$

for each  $k \in \mathbb{N}$ . As noted above, we may assume without loss of generality that  $\psi^k$  satisfies (2.2). Upon making this assumption, we have that the sequence  $(D\psi^k)_{k \in \mathbb{N}}$  is both equicontinuous and locally uniformly bounded on  $\mathbb{R}^m$ . By the Arzelà-Ascoli Theorem, there is a subsequence  $(\psi^{k_j})_{j \in \mathbb{N}}$  and  $\psi \in C^1(\mathbb{R}^m)$  such that  $\psi^{k_j} \rightarrow \psi$  and  $D\psi^{k_j} \rightarrow D\psi$  locally uniformly on  $\mathbb{R}^m$ . Moreover,  $\psi$  satisfies (1.2). Similarly, there is a subsequence  $(F^{k_j})_{j \in \mathbb{N}}$  and  $F \in C^1(\mathbb{M}^{m \times n})$  such that  $F^{k_j} \rightarrow F$  and  $DF^{k_j} \rightarrow DF$  locally uniformly on  $\mathbb{M}^{m \times n}$  and  $F$  satisfies (1.3).

Also note by (3.17) that there is  $\mathbf{v} \in L^2(U \times (0, T); \mathbb{R}^m)$  and a subsequence of  $(\mathbf{v}^{k_j})_{j \in \mathbb{N}}$  (not relabeled) such that

$$\mathbf{v}^{k_j} \rightharpoonup \mathbf{v}$$

in  $L^2((0, T); H^1(U; \mathbb{R}^m))$ . In view of Corollary 3.6,  $\mathbf{v}$  also satisfies (3.1). If we can establish (3.18) for each open  $V \subset\subset U$  and an interval  $[t_0, t_1] \subset (0, T)$ , we can then pass to the limit in

$$\int_0^T \int_U D\psi^k(\mathbf{v}^k) \cdot \mathbf{w}_t dx dt = \int_0^T \int_U DF^k(D\mathbf{v}^k) \cdot D\mathbf{w} dx dt, \quad (3.20)$$



as  $k = k_j \rightarrow \infty$  for each  $\mathbf{w} \in C_c^\infty(U \times (0, T); \mathbb{R}^m)$ . It would then follow that  $\mathbf{v}$  is necessarily a weak solution of (1.1) in  $U \times (0, T)$ . Thus, we focus on proving (3.18); along the way we will also verify (3.19). We finally note that since  $\partial U$  is smooth, it suffices to verify (3.18) for  $V$  with smooth boundary  $\partial V$ .

2. By Corollary 3.6 and our assumptions on  $\psi^k$  and  $F^k$ ,

$$\sup_{j \in \mathbb{N}} \left\{ \max_{t \in [t_0, t_1]} \int_V (|\mathbf{v}^{k_j}(x, t)|^2 + |D\mathbf{v}^{k_j}(x, t)|^2) dx + \int_{t_0}^{t_1} \int_V |\mathbf{v}_t^{k_j}|^2 dx dt \right\} < \infty. \quad (3.21)$$

This bound implies that  $\mathbf{v}^{k_j} : [t_0, t_1] \rightarrow L^2(V; \mathbb{R}^m)$  is uniformly equicontinuous and  $(\mathbf{v}^{k_j}(\cdot, t))_{j \in \mathbb{N}} \subset H^1(V; \mathbb{R}^m)$  is uniformly bounded independently of  $t \in [t_0, t_1]$ . Since  $H^1(V; \mathbb{R}^m) \subset L^2(V; \mathbb{R}^m)$  with compact embedding, there is a further subsequence (not relabeled) such that

$$\mathbf{v}^{k_j} \rightarrow \mathbf{v} \text{ in } C([t_0, t_1]; L^2(V; \mathbb{R}^m)). \quad (3.22)$$

This compactness is due to a well known result of J. P. Aubin [3]; see also [40] for an extended discussion.

Since  $(\mathbf{v}^{k_j}(\cdot, t))_{j \in \mathbb{N}}$  is bounded in  $H^1(V; \mathbb{R}^m)$  uniformly in  $t \in [t_0, t_1]$ , it follows from (3.22) that

$$D\mathbf{v}^{k_j}(\cdot, t) \rightharpoonup D\mathbf{v}(\cdot, t) \text{ in } L^2(V; \mathbb{M}^{m \times n}) \text{ uniformly for } t \in [t_0, t_1]. \quad (3.23)$$

We also have (up to a subsequence) that

$$DF(D\mathbf{v}^{k_j}) \rightharpoonup \xi \text{ in } L^2(V \times [t_0, t_1]; \mathbb{M}^{m \times n}).$$

Combined with (3.22), we can send  $k = k_j \rightarrow \infty$  in (3.20) to find

$$\partial_t(D\psi(\mathbf{v})) = \operatorname{div}(\xi). \quad (3.24)$$

3. We will now use the identity (2.1). Suppose  $\phi \in C_c^\infty(U \times (0, T))$  is supported in  $V \times (t_0, t_1)$  and is nonnegative. Then for  $t, s \in [t_0, t_1]$

$$\begin{aligned} & \int_U (\psi^{k_j})^* (D\psi^{k_j}(\mathbf{v}^{k_j}(x, t))) \phi(x, t) dx + \int_s^t \int_U \phi DF^{k_j}(D\mathbf{v}^{k_j}) \cdot D\mathbf{v}^{k_j} dx d\tau = \\ & \int_U (\psi^{k_j})^* (D\psi^{k_j}(\mathbf{v}^{k_j}(x, s))) \phi(x, s) dx + \int_s^t \int_U ((\psi^{k_j})^* (D\psi^{k_j}(\mathbf{v}^{k_j}))) \phi_t - \mathbf{v}^{k_j} \cdot DF^{k_j}(D\mathbf{v}^{k_j}) D\phi \, dx d\tau. \end{aligned}$$

Sending  $j \rightarrow \infty$  gives

$$\begin{aligned} & \int_U \psi^* (D\psi(\mathbf{v}(x, t))) \phi(x, t) dx + \lim_{j \rightarrow \infty} \int_s^t \int_U \phi DF^{k_j}(D\mathbf{v}^{k_j}) \cdot D\mathbf{v}^{k_j} dx d\tau = \\ & \int_U \psi^* (D\psi(\mathbf{v}(x, s))) \phi(x, s) dx + \int_s^t \int_U (\psi^* (D\psi(\mathbf{v}))) \phi_t - \mathbf{v} \cdot \xi D\phi \, dx d\tau. \end{aligned}$$

On the other hand, we can use equation (3.24) to derive the identity

$$\begin{aligned} & \int_U \psi^*(D\psi(\mathbf{v}(x, t))) \phi(x, t) dx + \int_s^t \int_U \phi \xi \cdot D\mathbf{v} dx d\tau = \\ & \int_U \psi^*(D\psi(\mathbf{v}(x, s))) \phi(x, s) dx + \int_s^t \int_U (\psi^*(D\psi(\mathbf{v})) \phi_t - \mathbf{v} \cdot \xi D\phi) dx d\tau. \end{aligned}$$

A proof of this identity can be made similar to the one given above for Proposition 3.5. Therefore,

$$\lim_{j \rightarrow \infty} \int_s^t \int_U \phi DF^{k_j}(D\mathbf{v}^{k_j}) \cdot D\mathbf{v}^{k_j} dx d\tau = \int_s^t \int_U \phi \xi \cdot D\mathbf{v} dx d\tau.$$

Consequently,

$$\begin{aligned} & \lim_{j \rightarrow \infty} \left\{ \lambda \int_s^t \int_V \phi |D\mathbf{v}^{k_j} - D\mathbf{v}|^2 dx d\tau \right\} \\ & \leq \lim_{j \rightarrow \infty} \int_s^t \int_V \phi (DF^{k_j}(D\mathbf{v}^{k_j}) - DF^{k_j}(D\mathbf{v})) \cdot (D\mathbf{v}^{k_j} - D\mathbf{v}) dx d\tau \\ & = 0. \end{aligned}$$

It follows that  $D\mathbf{v}^{k_j} \rightarrow D\mathbf{v}$  in  $L^2_{\text{loc}}(V \times (t_0, t_1); \mathbb{M}^{m \times n})$  and that  $\xi = DF(\mathbf{v})$ . In view of (3.24) and the arbitrariness of  $V$  and  $[t_0, t_1]$ , we conclude that  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ .

4. By (3.21) and (3.22),  $\mathbf{v}_t^{k_j} \rightharpoonup \mathbf{v}_t$  in  $L^2_{\text{loc}}(V \times (t_0, t_1); \mathbb{R}^m)$ . We have from part (ii) of Proposition 3.5 that

$$\begin{aligned} & \int_U \phi(x, t) F^{k_j}(D\mathbf{v}^{k_j}(x, t)) dx + \int_0^t \int_U \phi D^2\psi^{k_j}(\mathbf{v}^{k_j}) \mathbf{v}_t^{k_j} \cdot \mathbf{v}_t^{k_j} dx ds \\ & = \int_0^t \int_U \left( \phi_t F(D\mathbf{v}^{k_j}) - \mathbf{v}_t^{k_j} \cdot DF^{k_j}(D\mathbf{v}^{k_j}) D\phi \right) dx ds. \end{aligned} \quad (3.25)$$

for each nonnegative  $\phi \in C_c^\infty(U \times (0, T))$ . Letting  $j \rightarrow \infty$  gives

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int_U \phi(x, t) F^{k_j}(D\mathbf{v}^{k_j}(x, t)) dx + \liminf_{j \rightarrow \infty} \int_0^t \int_U \phi D^2\psi^{k_j}(\mathbf{v}^{k_j}) \mathbf{v}_t^{k_j} \cdot \mathbf{v}_t^{k_j} dx ds \\ & \leq \int_0^t \int_U (\phi_t F(D\mathbf{v}) - \mathbf{v}_t \cdot DF(D\mathbf{v}) D\phi) dx ds. \end{aligned}$$

On the other hand, since  $\mathbf{v}$  is a weak solution

$$\begin{aligned} & \int_U \phi(x, t) F(D\mathbf{v}(x, t)) dx + \int_0^t \int_U \phi D^2\psi(\mathbf{v}) \mathbf{v}_t \cdot \mathbf{v}_t dx ds \\ & = \int_0^t \int_U (\phi_t F(D\mathbf{v}) - \mathbf{v}_t \cdot DF(D\mathbf{v}) D\phi) dx ds. \end{aligned}$$

Therefore,

$$\liminf_{j \rightarrow \infty} \int_U \phi(x, t) F^{k_j}(D\mathbf{v}^{k_j}(x, t)) dx = \int_U \phi(x, t) F(D\mathbf{v}(x, t)) dx$$

and

$$\liminf_{j \rightarrow \infty} \int_0^t \int_U \phi D^2 \psi^{k_j}(\mathbf{v}^{k_j}) \mathbf{v}_t^{k_j} \cdot \mathbf{v}_t^{k_j} dx ds = \int_0^t \int_U \phi D^2 \psi(\mathbf{v}) \mathbf{v}_t \cdot \mathbf{v}_t dx ds.$$

for each  $t \in (0, T)$ . It is now routine to check using the uniform convexity of  $\psi^{k_j}$  and  $F^{k_j}$ , (3.22) and (3.23) that  $\mathbf{v}^{k_j}(\cdot, t) \rightarrow \mathbf{v}(\cdot, t)$  in  $H_{\text{loc}}^1(U; \mathbb{R}^m)$  for each  $t \in (0, T)$  and that  $\mathbf{v}_t^{k_j} \rightarrow \mathbf{v}_t$  in  $L_{\text{loc}}^2(U \times (0, T); \mathbb{R}^m)$ . In particular, we conclude (3.19).

5. With these strong convergence assertions and (3.25), we have that the sequence of functions

$$[0, T] \ni t \mapsto \int_U \phi(x, t) F^{k_j}(D\mathbf{v}^{k_j}(x, t)) dx$$

is uniformly equicontinuous. Indeed, the arguments we made above imply that

$$\begin{aligned} \frac{d}{dt} \int_U \phi F^{k_j}(D\mathbf{v}^{k_j}) dx &= \int_U \left( \phi_t F(D\mathbf{v}^{k_j}) - \mathbf{v}_t^{k_j} \cdot D F^{k_j}(D\mathbf{v}^{k_j}) D\phi \right) dx ds \\ &\quad - \int_U \phi D^2 \psi^{k_j}(\mathbf{v}^{k_j}) \mathbf{v}_t^{k_j} \cdot \mathbf{v}_t^{k_j} dx ds \end{aligned}$$

converges in  $L^1([0, T])$  and so is uniformly integrable. Combined with the pointwise convergence of  $\mathbf{v}^{k_j}(\cdot, t) \rightarrow \mathbf{v}(\cdot, t)$  in  $H_{\text{loc}}^1(U; \mathbb{R}^m)$ , we conclude

$$\lim_{j \rightarrow \infty} \int_U \phi(x, t) F^{k_j}(D\mathbf{v}^{k_j}(x, t)) dx = \int_U \phi(x, t) F(D\mathbf{v}(x, t)) dx$$

uniformly in  $[0, T]$ . Recalling (3.23) and that  $F$  is uniformly convex, we finally deduce (3.18).  $\square$

## 4 Partial regularity

In this section, we will complete the main goal of this work which is to verify Theorems 1 and 2. In order to establish Theorem 1, we will use the ideas that go into proving Proposition 3.9 to deduce a local Hölder regularity criterion for weak solutions. Then we will use a standard Poincaré inequality and Lebesgue differentiation to show almost everywhere Hölder regularity of the spatial gradient of weak solutions. As for Theorem 2, we will employ a more refined Poincaré inequality that is based on the fractional time differentiability of weak solutions that is asserted in Corollary 3.8.

## 4.1 A local regularity criterion

Let us denote a parabolic cylinder of radius  $r > 0$  centered at  $(x, t)$  as

$$Q_r(x, t) := B_r(x) \times (t - r^2/2, t + r^2/2)$$

and the average of  $\mathbf{w}$  over  $Q_r = Q_r(x, t)$  as

$$\mathbf{w}_{Q_r} := \iint_{Q_r} \mathbf{w} = \frac{1}{|Q_r|} \iint_{Q_r} \mathbf{w}(y, s) dy ds.$$

A quantity that will be of great utility to us is

$$E(x, t, r) := \iint_{Q_r} \left| \frac{\mathbf{v} - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)}{r} \right|^2 dy ds + \iint_{Q_r} |D\mathbf{v} - (D\mathbf{v})_{Q_r}|^2 dy ds.$$

Here  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$  and  $Q_r = Q_r(x, t) \subset U \times (0, T)$  with  $r > 0$ .

We will now derive an important decay property of  $E$ .

**Lemma 4.1.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ . For each  $L > 0$ , there are  $\epsilon, \vartheta, \rho \in (0, 1/2)$  for which*

$$\begin{cases} Q_r(x, t) \subset U \times (0, T), & r < \rho \\ |(\mathbf{v})_{Q_r}|, |(D\mathbf{v})_{Q_r}| \leq L \\ E(x, t, r) \leq \epsilon^2 \end{cases}$$

*implies*

$$E(x, t, \vartheta r) \leq \frac{1}{2} E(x, t, r).$$

*Proof.* 1. We will argue by contradiction. If the assertion is false, there is  $L_0 > 0$  and sequences  $(x_k, t_k) \in U \times (0, T)$ ,  $\epsilon_k \rightarrow 0$ ,  $\vartheta_k \equiv \vartheta \in (0, 1/2)$  (chosen below), and  $r_k \rightarrow 0$  as  $k \rightarrow +\infty$  such that

$$\begin{cases} Q_{r_k}(x_k, t_k) \subset U \times (0, T) \\ |(\mathbf{v})_{Q_{r_k}}|, |(D\mathbf{v})_{Q_{r_k}}| \leq L_0 \\ E(x_k, t_k, r_k) = \epsilon_k^2 \end{cases} \quad (4.1)$$

while

$$E(x_k, t_k, \vartheta r_k) > \frac{1}{2} \epsilon_k^2. \quad (4.2)$$

Define the sequence of mappings

$$\mathbf{v}^k(y, s) := \frac{\mathbf{v}(x_k + r_k y, t_k + r_k^2 s) - (\mathbf{v})_{Q_{r_k}} - (D\mathbf{v})_{Q_{r_k}} r_k y}{\epsilon_k r_k},$$

for  $(y, s) \in Q_1 := Q_1(0, 0)$  and  $k \in \mathbb{N}$ . As  $E(x_k, t_k, r_k) = \epsilon_k^2$ ,  $\mathbf{v}^k$  satisfies

$$\iint_{Q_1} |\mathbf{v}^k|^2 dy ds + \iint_{Q_1} |D\mathbf{v}^k|^2 dy ds = 1 \quad (4.3)$$

for each  $k \in \mathbb{N}$ . Moreover,  $\mathbf{v}^k$  is a weak solution of the PDE

$$\partial_s [D_w \psi^k(y, \mathbf{v}^k)] = \operatorname{div}_y [DF^k(D\mathbf{v}^k)] \quad (4.4)$$

in  $Q_1$ . Here

$$\psi^k(y, w) := \frac{\psi(a_k + r_k M_k y + \epsilon_k r_k w) - \psi(a_k + r_k M_k y) - D\psi(a_k + r_k M_k y) \cdot \epsilon_k r_k w}{\epsilon_k^2 r_k^2},$$

and

$$F^k(\xi) := \frac{F(M_k + \epsilon_k \xi) - F(M_k) - DF(M_k) \cdot \epsilon_k \xi}{\epsilon_k^2}.$$

We are now using the notation  $a_k := (\mathbf{v})_{Q_{r_k}} \in \mathbb{R}^m$ ,  $M_k := (D\mathbf{v})_{Q_{r_k}} \in \mathbb{M}^{m \times n}$ , and in view of (4.1), these sequences are bounded. Without any loss of generality, we assume  $a_k \rightarrow a \in \mathbb{R}^m$  and  $M_k \rightarrow M \in \mathbb{M}^{m \times n}$ .

2. Observe that for each  $y \in \mathbb{R}^n$ ,  $w \mapsto \psi^k(y, w)$  is uniformly convex and satisfies (1.2). Moreover,

$$\begin{cases} \psi^k(y, w) \rightarrow \frac{1}{2} D^2 \psi(a) w \cdot w \\ D_w \psi^k(y, w) \rightarrow D^2 \psi(a) w \end{cases}$$

for each  $(y, w) \in \mathbb{R}^n \times \mathbb{R}^m$  as  $k \rightarrow \infty$ . Likewise,  $F^k$  satisfies (1.2) and

$$\begin{cases} F^k(\xi) \rightarrow \frac{1}{2} D^2 F(M) \xi \cdot \xi \\ DF^k(\xi) \rightarrow D^2 F(M) \xi \end{cases}$$

for every  $\xi \in \mathbb{M}^{m \times n}$  as  $k \rightarrow \infty$ . Furthermore,  $\psi^k(y, \cdot)$  and  $F^k$  satisfies (2.2) for each  $y \in \mathbb{R}^n$ .

By (4.3) and a minor variant of Proposition 3.9, there is  $\mathbf{w} \in L^2([-1/2, 1/2]; H^1(B_1; \mathbb{R}^m))$  and a subsequence  $\{\mathbf{v}^{k_j}\}_{j \in \mathbb{N}}$  satisfying

$$\mathbf{v}^{k_j} \rightarrow \mathbf{w} \quad \text{in } C([t_0, t_1]; H^1(B_R; \mathbb{R}^m)) \quad (4.5)$$

for each  $[t_0, t_1] \subset (-1/2, 1/2)$  and  $R \in (0, 1)$ . Moreover,

$$\iint_{Q_1} |\mathbf{w}|^2 dy ds + \iint_{Q_1} |D\mathbf{w}|^2 dy ds = 1. \quad (4.6)$$

Recall that the weak formulation of (4.4) is

$$\iint_{Q_1} D\psi^k(y, \mathbf{v}^k) \cdot \phi_s dy ds = \iint_{Q_1} DF^k(D\mathbf{v}^k) \cdot D\phi dy ds$$

for each  $\phi \in C_c(Q_1; \mathbb{R}^m)$ . Passing to the limit as  $k = k_j \rightarrow \infty$  gives that  $\mathbf{w}$  is a weak solution of the linear evolution equation

$$\partial_s (D^2 \psi(a) \mathbf{w}) = \operatorname{div} (D^2 F(M) D\mathbf{w}) \quad (4.7)$$

in  $Q_1$  in the sense of Definition 3.1.

3. We claim that  $\mathbf{w} \in C^\infty(Q_1; \mathbb{R}^m)$ . First observe that by Corollaries 2.2 and 2.4,  $\mathbf{w}$  satisfies

$$\max_{|s| \leq \frac{1}{2}} \int_{B_1} \eta^2 |\mathbf{w}|^2 dy + \iint_{Q_1} \eta^2 |D\mathbf{w}|^2 dy ds \leq C \iint_{Q_1} (\eta |\eta_s| + |D\eta|^2) |\mathbf{w}|^2 dy ds$$

and

$$\max_{|s| \leq \frac{1}{2}} \int_{B_1} \eta^2 |D\mathbf{w}|^2 dy + \iint_{Q_1} \eta^2 |\mathbf{w}_s|^2 dy ds \leq C \iint_{Q_1} (\eta |\eta_s| + |D\eta|^2) |D\mathbf{w}|^2 dy ds$$

for each nonnegative  $\eta \in C_c^\infty(Q_1)$ . Here  $C = C(\lambda, \Lambda, \theta, \Theta)$ .

Let  $Q_R := Q_R(0, 0)$  for  $R \in (0, 1)$  and choose  $h$  so small that  $0 < |h| < 1 - R \leq \text{dist}(Q_R, \partial Q_1)$ . We define the time difference quotient

$$\partial_s^h \mathbf{w}(y, s) := \frac{\mathbf{w}(y, s + h) - \mathbf{w}(y, s)}{h}$$

for such values of  $h$  and  $(y, s) \in Q_R$ . Also note that  $\partial_s^h \mathbf{w}$  is a weak solution of the system (4.7) in  $Q_R$ . So as we observed above,  $\partial_s^h \mathbf{w}$  satisfies the energy estimates

$$\max_{|s| \leq \frac{1}{2} R^2} \int_{B_R} \eta^2 |\partial_s^h \mathbf{w}|^2 dy + \iint_{Q_R} \eta^2 |\partial_s^h D\mathbf{w}|^2 dy ds \leq C \iint_{Q_R} (\eta |\eta_s| + |D\eta|^2) |\partial_s^h \mathbf{w}|^2 dy ds$$

and

$$\max_{|s| \leq \frac{1}{2} R^2} \int_{B_R} \eta^2 |\partial_s^h D\mathbf{w}|^2 dy + \iint_{Q_R} \eta^2 |\partial_s^h \mathbf{w}_s|^2 dy ds \leq C \iint_{Q_R} (\eta |\eta_s| + |D\eta|^2) |\partial_s^h D\mathbf{w}|^2 dy ds$$

for  $0 < |h| < 1 - R$ . It follows that we have improved integrability of some of the derivatives of  $\mathbf{w}$ :  $\mathbf{w}_{ss} \in L_{\text{loc}}^2(Q_R; \mathbb{R}^m)$  and  $D\mathbf{w}_s \in L_{\text{loc}}^2(Q_R; \mathbb{M}^{m \times n})$  (Theorem 3, section 5.8.2 of [16]).

We can derive the same estimates for the spatial difference quotients

$$\partial_{y_i}^h \mathbf{w}(y, s) := \frac{\mathbf{w}(y + h e_i, s) - \mathbf{w}(y, s)}{h}$$

for  $(y, s) \in Q_R$ . Here  $i = 1, \dots, n$  and  $\{e_1, \dots, e_n\}$  is the standard basis in  $\mathbb{R}^n$ . Consequently, we can conclude  $D\mathbf{w}_{y_i} \in L_{\text{loc}}^2(Q_R; \mathbb{M}^{m \times n})$  for each  $i = 1, \dots, n$ . In particular, we have that each of the second derivatives of  $\mathbf{w}$  are locally square integrable on  $Q_1$ . Furthermore, we can proceed by induction to conclude that space-time derivatives of  $\mathbf{w}$  of all orders are locally square integrable on  $Q_1$  which implies that  $\mathbf{w} \in C^\infty(Q; \mathbb{R}^m)$ .

4. A close inspection of our justification that  $\mathbf{w}$  is smooth leads us to conclude that we can pointwise bound the higher order space-time derivatives of  $\mathbf{w}$  by the integral  $\iint_{Q_1} (|\mathbf{w}|^2 + |D\mathbf{w}|^2) dy ds$ . Using this fact and (4.6), we then conclude that there is a constant  $C_1$ , that depends only on  $n, \theta, \Theta, \lambda$  and  $\Lambda$ , such that

$$\iint_{Q_\vartheta} \left| \frac{\mathbf{w} - (\mathbf{w})_{Q_\vartheta} - (D\mathbf{w})_{Q_\vartheta} y}{\vartheta} \right|^2 dy ds + \iint_{Q_\vartheta} |D\mathbf{w} - (D\mathbf{w})_{Q_\vartheta}|^2 dy ds \leq C_1 \vartheta^2.$$

Here  $Q_\vartheta := Q_\vartheta(0, 0)$ . Let us now choose  $\vartheta_k \equiv \vartheta \in (0, 1/2)$  so small that  $C_1 \vartheta^2 \leq 1/4$ . In view of (4.5), we have for all sufficiently large  $j$

$$\iint_{Q_\vartheta} \left| \frac{\mathbf{v}^{k_j} - (\mathbf{v}^{k_j})_{Q_\vartheta} - (D\mathbf{v}^{k_j})_{Q_\vartheta} y}{\vartheta} \right|^2 dy ds + \iint_{Q_\vartheta} |D\mathbf{v}^{k_j} - (D\mathbf{v}^{k_j})_{Q_\vartheta}|^2 dy ds \leq \frac{3}{8}. \quad (4.8)$$

However, it is readily verified that inequality (4.2) implies that the left hand side of (4.8) is larger than  $1/2$  for all  $k \in \mathbb{N}$ . Therefore, we have the desired contradiction.  $\square$

We now seek to iterate the conclusion of Lemma 4.1. First let us recall a basic fact about the integral averages of  $\mathbf{v}$ . Observe for  $\tau \in (0, 1]$  and  $Q_r = Q_r(x, t) \subset U \times (0, T)$ ,

$$\begin{aligned} |(\mathbf{v})_{Q_{r\tau}} - (\mathbf{v})_{Q_r}| &= \left| \iint_{Q_{r\tau}} (\mathbf{v} - (\mathbf{v})_{Q_r}) dy ds \right| \\ &= \left| \iint_{Q_{r\tau}} (\mathbf{v} - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)) dy ds \right| \\ &\leq \left( \iint_{Q_{r\tau}} |\mathbf{v} - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)|^2 dy ds \right)^{1/2} \\ &\leq \frac{1}{\tau^{n/2+1}} \left( \iint_{Q_r} |\mathbf{v} - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)|^2 dy ds \right)^{1/2} \\ &= \frac{r}{\tau^{n/2+1}} \left( \iint_{Q_r} \left| \frac{\mathbf{v} - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)}{r} \right|^2 dy ds \right)^{1/2} \\ &\leq \frac{r}{\tau^{n/2+1}} E(x, t, r)^{1/2}. \end{aligned} \quad (4.9)$$

Similarly we have

$$\begin{aligned} |(D\mathbf{v})_{Q_{r\tau}} - (D\mathbf{v})_{Q_r}| &= \left| \iint_{Q_{r\tau}} (D\mathbf{v} - (D\mathbf{v})_{Q_r}) dy ds \right| \\ &\leq \left( \iint_{Q_{r\tau}} |D\mathbf{v} - (D\mathbf{v})_{Q_r}|^2 dy ds \right)^{1/2} \\ &\leq \frac{1}{\tau^{n/2+1}} \left( \iint_{Q_r} |D\mathbf{v} - (D\mathbf{v})_{Q_r}|^2 dy ds \right)^{1/2} \\ &\leq \frac{1}{\tau^{n/2+1}} E(x, t, r)^{1/2}. \end{aligned} \quad (4.10)$$

**Corollary 4.2.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ . Let  $L > 0$  and select  $\epsilon, \vartheta, \rho \in (0, 1/2)$  as in Lemma 4.1. If*

$$\begin{cases} Q_r(x, t) \subset U \times (0, T), & r < \rho \\ |(\mathbf{v})_{Q_r}|, |(D\mathbf{v})_{Q_r}| < \frac{1}{2}L \\ E(x, t, r) < \epsilon_1^2 \end{cases}, \quad (4.11)$$

where  $\epsilon_1 := \min \left\{ \epsilon, \frac{\vartheta^{n/2+1}}{6} L \right\}$ , then

$$\begin{cases} |(\mathbf{v})_{Q_{\vartheta^k r}}|, |(D\mathbf{v})_{Q_{\vartheta^k r}}| < L \\ E(x, t, \vartheta^k r) \leq \frac{1}{2^k} E(x, t, r) \end{cases} \quad (4.12)$$

for each  $k \in \mathbb{N}$ .

*Proof.* We will use mathematical induction on  $k$ . Let us first consider the base case  $k = 1$ . By Lemma 4.1 and (4.11), we have  $E(x, t, \vartheta r) \leq \frac{1}{2} E(x, t, r)$ . We also have from (4.9) that

$$\begin{aligned} |(\mathbf{v})_{Q_{\vartheta r}}| &\leq |(\mathbf{v})_{Q_{\vartheta r}} - (\mathbf{v})_{Q_r}| + |(\mathbf{v})_{Q_r}| \\ &\leq \frac{r}{\vartheta^{n/2+1}} E(x, t, r)^{1/2} + |(\mathbf{v})_{Q_r}| \\ &\leq \frac{\epsilon_1}{\vartheta^{n/2+1}} + \frac{1}{2} L \\ &< L. \end{aligned}$$

Similarly, we can employ (4.10) to conclude  $|(D\mathbf{v})_{Q_{\vartheta r}}| < L$ . Therefore, we have established (4.12) for  $k = 1$ .

Now suppose that (4.12) holds for  $k = 1, 2, \dots, j \geq 1$ . In view of (4.9), we can employ the triangle inequality to deduce

$$\begin{aligned} |(\mathbf{v})_{Q_{\vartheta^{j+1} r}}| &\leq \sum_{k=0}^{j-1} |(\mathbf{v})_{Q_{\vartheta^{k+1} r}} - (\mathbf{v})_{Q_{\vartheta^k r}}| + |(\mathbf{v})_{Q_r}| \\ &\leq \sum_{k=1}^{j-1} \frac{r}{\vartheta^{n/2+1}} E(x, t, \vartheta^k r)^{1/2} + |(\mathbf{v})_{Q_r}| \\ &< \sum_{k=1}^{j-1} \frac{1}{\vartheta^{n/2+1}} \frac{1}{\sqrt{2}^k} E(x, t, r)^{1/2} + \frac{1}{2} L \\ &\leq \frac{\epsilon_1}{\vartheta^{n/2+1}} \sum_{k=1}^{j-1} \frac{1}{\sqrt{2}^k} + \frac{1}{2} L \\ &\leq \frac{\epsilon_1}{\vartheta^{n/2+1}} \sum_{k=1}^{\infty} \left( \frac{3}{4} \right)^k + \frac{1}{2} L \\ &\leq \frac{3\epsilon_1}{\vartheta^{n/2+1}} + \frac{1}{2} L \\ &\leq L. \end{aligned}$$

In nearly the same fashion, we can use (4.10) to deduce  $|(D\mathbf{v})_{Q_{\vartheta^{j+1} r}}| < L$ . As

$$E(x, t, \vartheta^j r) \leq \frac{1}{2^j} E(x, t, r) \leq \frac{1}{2^j} \epsilon_1^2 < \epsilon^2,$$



Lemma 4.1 then gives

$$E(x, t, \vartheta^{j+1}r) = E(x, t, \vartheta(\vartheta^j r)) \leq \frac{1}{2}E(x, t, \vartheta^j r) \leq \frac{1}{2^{j+1}}E(x, t, r).$$

□

**Corollary 4.3.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) in  $U \times (0, T)$ . Let  $L > 0$  and suppose there are  $(x, t) \in U \times (0, T)$  and  $r > 0$  as in (4.11). Then there exist  $C \geq 0$ ,  $\rho_1 \in (0, \rho)$ ,  $\alpha \in (0, 1)$  and a neighborhood  $O \subset U \times (0, T)$  of  $(x, t)$  such that*

$$E(y, s, R) \leq CR^\alpha, \quad R \in (0, \rho_1), \quad (y, s) \in O.$$

*Proof.* Let  $R \in (0, r)$  and choose  $k \in \mathbb{N}$  such that  $\vartheta^{k+1}r < R \leq \vartheta^k r$ . For  $f \in L^2_{\text{loc}}((0, T); H^1_{\text{loc}}(U))$ , we have

$$\iint_{Q_R} |f - f_{Q_R}|^2 dy ds \leq \frac{4}{\vartheta^{n+2}} \iint_{Q_{\vartheta^k r}} |f - f_{Q_{\vartheta^k r}}|^2 dy ds \quad (4.13)$$

and

$$\begin{aligned} & \iint_{Q_R} \left| \frac{f - f_{Q_R} - (Df)_{Q_R} \cdot (y - x)}{R} \right|^2 dy ds \leq \\ & \frac{4}{\vartheta^{2(n+3)}} \left\{ \iint_{Q_{\vartheta^k r}} \left| \frac{f - f_{Q_{\vartheta^k r}} - (Df)_{Q_{\vartheta^k r}} \cdot (y - x)}{\vartheta^k r} \right|^2 dy ds + \iint_{Q_{\vartheta^k r}} |Df - (Df)_{Q_{\vartheta^k r}}|^2 dy ds \right\} \end{aligned} \quad (4.14)$$

These inequalities can be derived similarly as we did for (4.9) and (4.10); they are also proved in Corollary 4.9 of [26].

Letting  $f = v^i$  in (4.14) and  $f = Dv^i$  in (4.13) and summing over  $i = 1, \dots, m$  gives

$$E(x, t, R) \leq \frac{8}{\vartheta^{2(n+3)}} E(x, t, \vartheta^k r).$$

In view of Corollary 4.2,

$$\begin{aligned} E(x, t, R) & \leq \frac{8}{\vartheta^{2(n+3)}} \frac{1}{2^k} E(x, t, r) \\ & \leq \frac{8\epsilon_1^2}{\vartheta^{2(n+3)}} \frac{1}{2^k} \\ & = \frac{16\epsilon_1^2}{\vartheta^{2(n+3)}} e^{-(k+1)\log 2} \\ & \leq \frac{16\epsilon_1^2}{\vartheta^{2(n+3)}} \left( \frac{R}{r} \right)^{\frac{\ln(1/2)}{\ln \vartheta}}. \end{aligned}$$

Recall that  $(\mathbf{v})_{Q_r(y,s)}$ ,  $(D\mathbf{v})_{Q_r(y,s)}$ , and  $E(y, s, r)$  are all continuous functions of  $(y, s) \in U \times (0, T)$  and  $r > 0$ . Therefore, there exists  $\rho_1, \rho_2 \in (0, \rho)$  and a neighborhood  $O$  of  $(x, t)$

such that (4.11) holds for all  $(y, s) \in O$  and  $r \in (\rho_1, \rho_2)$ . As a result, we can repeat the same computation above to conclude

$$E(y, s, R) \leq \frac{16\epsilon_1^2}{\vartheta^{2(n+3)}} \left( \frac{R}{\rho_1} \right)^{\frac{\ln(1/2)}{\ln \vartheta}}$$

for  $(y, s) \in O$  and  $R \in (0, \rho_1)$ . □

## 4.2 Partial regularity

We are finally in position to prove Theorem 1 and 2. We will start with Theorem 1, which asserts the almost everywhere Hölder continuity of weak solutions.

*Proof of Theorem 1.* We first claim that the set of points  $(x, t)$  for which the following limits hold

$$\begin{cases} \lim_{r \rightarrow 0^+} (\mathbf{v})_{Q_r(x, t)} = \mathbf{v}(x, t) \\ \lim_{r \rightarrow 0^+} (D\mathbf{v})_{Q_r(x, t)} = D\mathbf{v}(x, t) \\ \lim_{r \rightarrow 0^+} E(x, t, r) = 0 \end{cases}$$

has full Lebesgue measure in  $U \times (0, T)$ . This is evident for the first two limits by a version of Lebesgue's differentiation theorem for parabolic cylinders [27].

As for the third limit, recall Poincaré's inequality on a cylinder  $Q_r = Q_r(x, t) \subset U \times (0, T)$ : there is a constant  $C_0$  such that

$$\iint_{Q_r} |\mathbf{w} - (\mathbf{w})_{Q_r}|^2 dy ds \leq C_0 \left\{ r^4 \iint_{Q_r} |\mathbf{w}_t|^2 dy ds + r^2 \iint_{Q_r} |D\mathbf{w}|^2 dy ds \right\}$$

for each  $\mathbf{w} \in H_{\text{loc}}^1(U \times (0, T); \mathbb{R}^m)$ . Substituting

$$\mathbf{w}(y, s) := \mathbf{v}(y, s) - (\mathbf{v})_{Q_r} - (D\mathbf{v})_{Q_r}(y - x)$$

in the Poincaré inequality above gives

$$E(x, t, r) \leq (C_0 + 1) \left\{ r^2 \iint_{Q_r} |\mathbf{v}_t|^2 dy ds + \iint_{Q_r} |D\mathbf{v} - (D\mathbf{v})_{Q_r}|^2 dy ds \right\}. \quad (4.15)$$

Again we invoke Lebesgue differentiation [27] to conclude  $\lim_{r \rightarrow 0^+} E(x, t, r) = 0$  on a set of full Lebesgue measure.

It now follows that for Lebesgue almost every  $(x, t) \in U \times (0, T)$  there are  $L, r > 0$  such that (4.11) holds. At any such  $(x, t) \in U \times (0, T)$ , we can then apply the conclusion of Corollary 4.3. In particular, there is a neighborhood  $O \subset U \times (0, T)$  and  $\rho_1 > 0$  such that  $E(y, s, R) \leq CR^\alpha$  for  $Q_R = Q_R(y, s) \subset U \times (0, T)$  with  $(y, s) \in O$  and  $R < \rho_1$ . Therefore,

$$\left( \iint_{Q_R} |D\mathbf{v} - (D\mathbf{v})_{Q_R}|^2 dy ds \right)^{1/2} \leq E(y, s, R)^{1/2} \leq \sqrt{C} R^{\frac{1}{2}\alpha}.$$

By Campanato's criterion [5, 10],  $D\mathbf{v}$  is Hölder continuous in a neighborhood of  $(x, t)$ . Consequently, the set  $\mathcal{O}$  of points  $(x, t)$  for which there is a neighborhood of  $(x, t)$  such that  $D\mathbf{v}$  is Hölder continuous has full Lebesgue measure. By definition,  $\mathcal{O}$  is open. This concludes a proof of Theorem 1.  $\square$

*Remark 4.4.* The above argument also shows that  $\mathbf{v}$  is Hölder continuous in a neighborhood of  $(x, t)$ . Indeed

$$\begin{aligned} \left( \iint_{Q_R} |\mathbf{v} - (\mathbf{v})_{Q_R}|^2 dz d\tau \right)^{1/2} &\leq \left( \iint_{Q_R} |\mathbf{v} - (\mathbf{v})_{Q_R} - (D\mathbf{v})_{Q_R}(z - y)|^2 dz d\tau \right)^{1/2} + |(D\mathbf{v})_{Q_R}|R \\ &= RE(y, s, R)^{1/2} + LR \\ &\leq \sqrt{C}R^{1+\frac{1}{2}\alpha} + LR \\ &\leq (\sqrt{C}\rho_1^{\frac{1}{2}\alpha} + L)R. \end{aligned}$$

Let us recall the definition of parabolic Hausdorff measure.

**Definition 4.5.** For  $G \subset \mathbb{R}^n \times \mathbb{R}$ ,  $s \in [0, n + 2]$ ,  $\delta > 0$ , set

$$\mathcal{P}_\delta^s(G) := \inf \left\{ \sum_{i \in \mathbb{N}} r_i^s : G \subset \bigcup_{i \in \mathbb{N}} Q_{r_i}(x_i, t_i), r_i \leq \delta \right\}.$$

The  $s$ -dimensional parabolic Hausdorff measure of  $G$  is defined

$$\mathcal{P}^s(G) := \sup_{\delta > 0} \mathcal{P}_\delta^s(G).$$

Moreover, the parabolic Hausdorff dimension of  $G$  is the number

$$\dim_{\mathcal{P}}(G) := \inf \{s \geq 0 : \mathcal{P}^s(G) = 0\}.$$

We note that  $\mathcal{P}^s$  is an outer measure on  $\mathbb{R}^n \times \mathbb{R}$  for each  $s \in [0, n + 2]$ . While there are many important properties of (general) Hausdorff measure (as detailed in [17] and [37]), we will only make use of one fact about parabolic Hausdorff measure that is based on a Poincaré inequality for fractionally differentiable functions. The Poincaré inequality we have in mind is as follows, we will omit a proof of this inequality as it is stated and proved in Lemma 2.16 of [15].

**Lemma 4.6.** Let  $\gamma \in (0, 1)$ . Suppose  $w \in L_{loc}^2(U \times (0, T))$  satisfies

$$\int_{t_0}^{t_1} \int_V \int_V \frac{|w(x, t) - w(y, t)|^2}{|x - y|^{n+2\gamma}} dx dy dt + \int_{t_0}^{t_1} \int_{t_0}^{t_1} \int_V \frac{|w(x, t) - w(x, s)|^2}{|t - s|^{1+\gamma}} dx dt ds < \infty \quad (4.16)$$

for each open  $V \subset\subset U$  and interval  $[t_0, t_1] \subset (0, T)$ . Then there is a constant  $C_0$  such that for every  $Q_r = Q_r(x, t) \subset U \times (0, T)$ ,

$$\iint_{Q_r} |w - (w)_{Q_r}|^2 dy ds \leq C_0 r^{2\gamma} \left\{ \int_{t-r^2/2}^{t+r^2/2} \int_{B_r(x)} \int_{B_r(x)} \frac{|w(x, s) - w(y, s)|^2}{|x - y|^{n+2\gamma}} dx dy ds \right.$$

$$+ \int_{t-r^2/2}^{t+r^2/2} \int_{t-r^2/2}^{t+r^2/2} \int_{B_r(x)} \frac{|w(y, \tau) - w(y, s)|^2}{|\tau - s|^{1+\gamma}} dy d\tau ds \Big\}.$$

The crucial fact about parabolic Hausdorff measure is as follows. Versions of this assertion which can be found in Proposition 3.3 of [14], Proposition 4.2 in [31], Theorem 3 in section 2.4.3 of [17], so we will not provide a proof.

**Proposition 4.7.** *Let  $\gamma \in (0, 1)$ , and suppose  $w \in L_{loc}^2(U \times (0, T))$  satisfies (4.16). Then*

$$\dim_{\mathcal{P}} \left( \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} \iint_{Q_r(x, t)} |w - (w)_{Q_r(x, t)}|^2 dy ds > 0 \right\} \right) \leq n + 2 - 2\gamma$$

and

$$\dim_{\mathcal{P}} \left( \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} |(w)_{Q_r(x, t)}| = +\infty \right\} \right) \leq n + 2 - 2\gamma.$$

We will combine these facts with our previous estimates to fashion a fairly simple proof of Theorem 2. Before proceeding to the proof, we will need to verify a local version of inequality (2.6).

**Lemma 4.8.** *Assume  $\mathbf{v}$  is a weak solution of (1.1) on  $U \times (0, T)$ . There is a constant  $C$  depending only on  $\theta, \lambda, \Theta$ , and  $\Lambda$  such that*

$$\iint_{Q_r(x, t)} |\mathbf{v}_t|^2 dy ds \leq \frac{C}{r^2} \iint_{Q_{2r}(x, t)} |D\mathbf{v} - (D\mathbf{v})_{Q_{2r}(x, t)}|^2 dy ds \quad (4.17)$$

whenever  $Q_{2r}(x, t) \subset U \times (0, T)$ .

*Proof.* Fix  $A \in \mathbb{M}^{m \times n}$  and  $\phi \in C_c^\infty(U \times (0, T))$ . As we computed (2.5), we find

$$\begin{aligned} & \frac{d}{dt} \int_U \phi (F(D\mathbf{v}) - DF(A) \cdot (DV - A)) dx + \int_U \phi \partial_t (D\psi(\mathbf{v})) \cdot \mathbf{v}_t dx \\ &= \int_U (\phi_t (F(D\mathbf{v}) - DF(A) \cdot (DV - A)) - \mathbf{v}_t \cdot (DF(D\mathbf{v}) - DF(A)) D\phi) dx \end{aligned}$$

for almost every  $t \in (0, T)$ . And setting  $\phi = \eta^2$  for  $\eta \geq 0$  gives

$$\max_{0 \leq t \leq T} \int_U \eta^2 |D\mathbf{v} - A|^2 dx + \int_0^T \int_U \eta^2 |\mathbf{v}_t|^2 dx dt \leq C \int_0^T \int_U (\eta |\eta_t| + |D\eta|^2) |D\mathbf{v} - A|^2 dx dt \quad (4.18)$$

the same way that we derived (2.6).

Now let  $\eta_0 \in C_c^\infty(\mathbb{R}^n)$  satisfy

$$\begin{cases} 0 \leq \eta_0 \leq 1 \\ \eta_0 \equiv 1 \text{ in } B_r(x) \\ \eta_0 \equiv 0 \text{ in } \mathbb{R}^n \setminus B_{2r}(x) \\ |D\eta_0| \leq 2/r \end{cases}$$

and  $\eta_1 \in C_c^\infty(\mathbb{R})$  satisfy

$$\begin{cases} 0 \leq \eta_1 \leq 1 \\ \eta_1 \equiv 1 & \text{in } (t - r^2/2, t + r^2/2) \\ \eta_1 \equiv 0 & \text{in } \mathbb{R} \setminus (t - 2r^2, t + 2r^2) \\ |\partial_\tau \eta_1| \leq 2/r^2. \end{cases}$$

We conclude (4.17) by choosing  $\eta(y, \tau) = \eta_0(y) \cdot \eta_1(\tau)$  and  $A = (D\mathbf{v})_{Q_{2r}(x,t)}$  in (4.18).  $\square$

*Proof of Theorem 2.* Choose  $\beta$  as in (3.15) and select  $\epsilon > 0$  so small that

$$\beta + \epsilon \in \left(0, \frac{1}{2} - \frac{1}{p}\right).$$

Let us also recall our definition of  $\mathcal{O}$

$$\mathcal{O} = \{(x, t) \in U \times (0, T) : D\mathbf{v} \text{ is Hölder continuous in some neighborhood of } (x, t)\}.$$

By Corollary 4.3,

$$U \times (0, T) \setminus \mathcal{O} \subset G_1 \cup G_2 \cup G_3.$$

Here

$$\begin{aligned} G_1 &= \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} E(x, t, r) > 0 \right\}, \\ G_2 &= \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} |(\mathbf{v})_{Q_r(x,t)}| = +\infty \right\}, \end{aligned}$$

and

$$G_3 = \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} |(D\mathbf{v})_{Q_r(x,t)}| = +\infty \right\}.$$

It suffices to show  $\mathcal{P}^{n+2-2\beta}(G_i) = 0$  for  $i = 1, 2, 3$ .

Observe by (4.15) and (4.17)

$$\limsup_{r \rightarrow 0^+} E(x, t, r) \leq 2(C_0 + 1) \limsup_{r \rightarrow 0^+} \iint_{Q_r(x,t)} |D\mathbf{v} - (D\mathbf{v})_{Q_r(x,t)}|^2 dy ds$$

for any  $(x, t) \in U \times (0, T)$ . By Lemma 3.3 and Corollary 3.8,  $w = v_{x_j}^i$  satisfies (4.16) with  $\gamma = \beta + \epsilon$  for each  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Here we are using the inclusion  $H_{\text{loc}}^1(U) \subset H_{\text{loc}}^\sigma(U)$  ( $0 < \sigma < 1$ ), which is proved in Proposition 2.2 of [12]. Therefore,

$$G_1 \subset \left\{ (x, t) \in U \times (0, T) : \limsup_{r \rightarrow 0^+} \iint_{Q_r(x,t)} |D\mathbf{v} - (D\mathbf{v})_{Q_r(x,t)}|^2 dy ds > 0 \right\}.$$

In view of Proposition 4.7, we conclude

$$\dim_{\mathcal{P}}(G_1) \leq n + 2 - 2(\beta + \epsilon) < n + 2 - 2\beta.$$

It follows that  $\mathcal{P}^{n+2-2\beta}(G_1) = 0$ . Likewise, we can also conclude  $\mathcal{P}^{n+2-2\beta}(G_3) = 0$ . The conclusion  $\mathcal{P}^{n+2-2\beta}(G_2) = 0$  follows similarly as  $v^i$  satisfies (4.16) for every  $\gamma \in (0, 1)$  and  $i = 1, \dots, m$ .  $\square$

## A Dirichlet problem

Assume  $\psi \in C^2(\mathbb{R}^m)$  and  $F \in C^2(\mathbb{M}^{m \times n})$  satisfy (1.2), (1.3) and (2.2). For a given  $\mathbf{g} \in H^1(U; \mathbb{R}^m)$ , we will show a that weak solution of the initial value problem

$$\begin{cases} \partial_t (D\psi(\mathbf{v})) = \operatorname{div} DF(D\mathbf{v}), & \text{in } U \times (0, T) \\ \mathbf{v} = 0, & \text{on } \partial U \times [0, T) \\ \mathbf{v} = \mathbf{g}, & \text{on } U \times \{0\} \end{cases} \quad (\text{A.1})$$

exists. In particular, the solution  $\mathbf{v}$  we construct solution will also be a weak solution of (1.1). We also acknowledge that the existence of a solution to (A.1) has already been established in various contexts such as [1, 13, 46]. We have added this appendix because we will require more integrability of our weak solutions (see (A.6) below).

Note that any smooth solution  $\mathbf{v}$  of (A.1) satisfies

$$\frac{d}{dt} \int_U \psi^*(D\psi(\mathbf{v})) dx = - \int_U DF(D\mathbf{v}) \cdot D\mathbf{v} dx$$

and therefore

$$\int_U \psi^*(D\psi(\mathbf{v}(x, t))) dx + \int_0^t \int_U DF(D\mathbf{v}) \cdot D\mathbf{v} dx ds = \int_U \psi^*(D\psi(\mathbf{g})) dx \quad (\text{A.2})$$

for each  $t \geq 0$ . It follows that

$$\sup_{0 \leq t \leq T} \int_U |\mathbf{v}(x, t)|^2 dx + \int_0^T \int_U |D\mathbf{v}|^2 dx dt \leq C \int_U |\mathbf{g}|^2 dx \quad (\text{A.3})$$

for a constant  $C$  depending only on  $\theta, \lambda, \Theta$  and  $\Lambda$ .

The identity

$$\frac{d}{dt} \int_U F(D\mathbf{v}) dx = - \int_U D^2\psi(\mathbf{v}) \mathbf{v}_t \cdot \mathbf{v}_t dx$$

also holds for any smooth solution  $\mathbf{v}$  of (A.1). Integrating this identity gives

$$\int_U F(D(\mathbf{v}(x, t))) dx + \int_0^t \int_U D^2\psi(\mathbf{v}) \mathbf{v}_t \cdot \mathbf{v}_t dx ds = \int_U F(D\mathbf{g}) dx \quad (\text{A.4})$$

for each  $t \geq 0$ . Using standard manipulations, we can also deduce

$$\sup_{0 \leq t \leq T} \int_U |D\mathbf{v}(x, t)|^2 dx + \int_0^T \int_U |\mathbf{v}_t|^2 dx ds \leq C \int_U |D\mathbf{g}|^2 dx \quad (\text{A.5})$$

for some constant  $C$  only depending on  $\theta, \Theta, \lambda$  and  $\Lambda$ .

Inequalities (A.3) and (A.5) leads us to define weak solutions analogously to Definition 3.1.

**Definition A.1.** Suppose  $\mathbf{g} \in H^1(U; \mathbb{R}^m)$ . A measurable mapping  $\mathbf{v} : U \times [0, T] \rightarrow \mathbb{R}^m$  is a *weak solution* of (A.1) if  $\mathbf{v}$  satisfies: (i)

$$\mathbf{v} \in L^\infty([0, T]; H_0^1(U; \mathbb{R}^m)) \quad \text{and} \quad \mathbf{v}_t \in L^2(U \times [0, T]; \mathbb{R}^m); \quad (\text{A.6})$$

(ii)

$$\int_0^T \int_U D\psi(\mathbf{v}) \cdot \mathbf{w}_t dx dt = \int_0^T \int_U DF(D\mathbf{v}) \cdot D\mathbf{w} dx dt, \quad (\text{A.7})$$

for all  $\mathbf{w} \in C_c^\infty(U \times (0, T); \mathbb{R}^m)$ ; and (iii)

$$\mathbf{v}(\cdot, 0) = \mathbf{g}. \quad (\text{A.8})$$

As we argued for weak solutions of (1.1), we can deduce that (A.2) and (A.4) hold for weak solutions of (A.1). It follows of course that (A.3) and (A.5) also are valid for weak solutions of (A.1). Moreover, we can verify

$$\mathbf{v} \in C([0, T]; H_0^1(U; \mathbb{R}^m))$$

similar to how we argued in part 4 of Proposition 3.5. Therefore, it makes sense to prescribe the initial condition (A.8).

One way of constructing a weak solution to (A.1) is to use the following implicit time scheme. Let  $N \in \mathbb{N}$ , set  $\tau = T/N$  and  $\mathbf{v}^0 = \mathbf{g}$ , then find  $\mathbf{v}^k \in H_0^1(U; \mathbb{R}^m)$  that satisfies

$$\begin{cases} \frac{D\psi(\mathbf{v}^k) - D\psi(\mathbf{v}^{k-1})}{\tau} = \operatorname{div} DF(D\mathbf{v}^k), & \text{in } U \\ \mathbf{v}^k = 0, & \text{on } \partial U \end{cases} \quad (\text{A.9})$$

weakly for  $k = 1, \dots, N$ . That is,

$$\int_U \frac{D\psi(\mathbf{v}^k) - D\psi(\mathbf{v}^{k-1})}{\tau} \cdot \mathbf{w} dx + \int_U DF(D\mathbf{v}^k) \cdot D\mathbf{w} dx = 0 \quad (\text{A.10})$$

for each  $\mathbf{w} \in H_0^1(U; \mathbb{R}^m)$ . In particular, once  $\mathbf{v}^1, \dots, \mathbf{v}^j$  are found, then  $\mathbf{v}^{j+1}$  is the unique minimizer of the strictly convex and coercive functional

$$H_0^1(U; \mathbb{R}^m) \ni \mathbf{u} \mapsto \int_U \left( F(D\mathbf{u}) + \frac{\psi(\mathbf{u}) - D\psi(\mathbf{v}^j) \cdot \mathbf{u}}{\tau} \right) dx.$$

Therefore, (A.1) has a unique solution  $\{\mathbf{v}^1, \dots, \mathbf{v}^N\} \subset H_0^1(U; \mathbb{R}^m)$ .

Let us now derive a few estimates on solutions of the implicit time scheme analogous to (A.3) and (A.5).

**Lemma A.2.** Suppose  $\{\mathbf{v}^1, \dots, \mathbf{v}^N\} \subset H_0^1(U; \mathbb{R}^m)$  is a solution of the implicit time scheme (A.9) with  $\mathbf{v}^0 = \mathbf{g}$ . Then there is a constant  $C$  such that

$$\max_{1 \leq k \leq N} \int_U |\mathbf{v}^k|^2 dx + \tau \sum_{k=1}^N \int_U |D\mathbf{v}^k|^2 dx \leq C \int_U |\mathbf{g}|^2 dx \quad (\text{A.11})$$

and

$$\max_{1 \leq k \leq N} \int_U |D\mathbf{v}^k|^2 dx + \tau \sum_{k=1}^N \int_U \left| \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\tau} \right|^2 dx \leq C \int_U |D\mathbf{g}|^2 dx. \quad (\text{A.12})$$

*Proof.* Choosing  $\mathbf{w} = \mathbf{v}^k$  in (A.10) gives

$$\begin{aligned} \int_U DF(D\mathbf{v}^k) \cdot D\mathbf{v}^k dx &= \int_U \frac{D\psi(\mathbf{v}^{k-1}) - D\psi(\mathbf{v}^k)}{\tau} \cdot \mathbf{v}^k dx \\ &= \int_U \frac{D\psi^*(D\psi(\mathbf{v}^k)) \cdot (D\psi(\mathbf{v}^{k-1}) - D\psi(\mathbf{v}^k))}{\tau} dx \\ &\leq \int_U \frac{(\psi^*(D\psi(\mathbf{v}^{k-1})) - \psi^*(D\psi(\mathbf{v}^k)))}{\tau} dx. \end{aligned}$$

Summing over  $k = 1, \dots, j \leq N$  we find

$$\int_U \psi^*(D\psi(\mathbf{v}^j)) dx + \tau \sum_{k=1}^j \int_U DF(D\mathbf{v}^k) \cdot D\mathbf{v}^k dx \leq \int_U \psi^*(D\psi(\mathbf{g})) dx.$$

Consequently, we can now employ elementary manipulations to derive (A.11).

Selecting  $\mathbf{w} = \mathbf{v}^k - \mathbf{v}^{k-1}$  in (A.10) gives

$$\begin{aligned} \int_U \frac{D\psi(\mathbf{v}^k) - D\psi(\mathbf{v}^{k-1})}{\tau} \cdot \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\tau} dx &= - \int_U DF(D\mathbf{v}^k) \cdot \frac{D\mathbf{v}^k - D\mathbf{v}^{k-1}}{\tau} dx \\ &\leq \int_U \frac{F(D\mathbf{v}^{k-1}) - F(D\mathbf{v}^k)}{\tau} dx. \end{aligned}$$

Summing over  $k = 1, \dots, j \leq N$  gives

$$\int_U F(D\mathbf{v}^j) dx + \tau \sum_{k=1}^j \int_U \frac{D\psi(\mathbf{v}^k) - D\psi(\mathbf{v}^{k-1})}{\tau} \cdot \frac{\mathbf{v}^k - \mathbf{v}^{k-1}}{\tau} dx \leq \int_U F(D\mathbf{g}) dx.$$

It is now routine to conclude (A.12).  $\square$

We will now use (A.11) and (A.12) to show that (A.1) has a weak solution. Consequently, there is at least one weak solution of (1.1) in  $U \times (0, T)$  that is partially regular as described in Theorem 1 and 2.

**Proposition A.3.** *There exists a weak solution  $\mathbf{v}$  of (A.1).*

*Proof.* Let  $N \in \mathbb{N}$ ,  $\tau = T/N$ , and suppose  $\{\mathbf{v}^1, \dots, \mathbf{v}^N\} \subset H_0^1(U; \mathbb{R}^m)$  is a solution of the implicit time scheme (A.9) with  $\mathbf{v}^0 = \mathbf{g} \in H^1(U; \mathbb{R}^m)$ . Set  $\tau_k := \tau k$  for  $k = 0, \dots, N$  and



define

$$\begin{cases} \mathbf{v}^N(\cdot, t) := \begin{cases} \mathbf{g}, & t = 0 \\ \mathbf{v}^k, & \tau_{k-1} < t \leq \tau_k \end{cases} \\ \mathbf{w}^N(\cdot, t) := D\psi(\mathbf{v}^{k-1}) + \left(\frac{t - \tau_k}{\tau}\right) (D\psi(\mathbf{v}^k) - D\psi(\mathbf{v}^{k-1})), & \tau_{k-1} \leq t \leq \tau_k \end{cases}$$

for  $t \in [0, T]$ . It is straightforward to use (A.9) and check that

$$\|\partial_t(\mathbf{w}^N)\|_{H^{-1}(U; \mathbb{R}^m)} = \|DF(D\mathbf{v}^N)\|_{L^2(U; \mathbb{M}^{m \times n})}$$

holds for  $t \in (0, T) \setminus \{\tau_1, \dots, \tau_N\}$ .

Furthermore, it is routine to use (A.11) to show

$$\sup_{0 \leq t \leq T} \int_U |\mathbf{v}^N(x, t)|^2 dx + \int_0^T \int_U |D\mathbf{v}^N|^2 dx dt \leq C \int_U |\mathbf{g}|^2 dx,$$

which in turn implies that  $\mathbf{w}^N$  fulfills

$$\sup_{0 \leq t \leq T} \int_U |\mathbf{w}^N(x, t)|^2 dx + \int_0^T \|\partial_t(\mathbf{w}^N(\cdot, t))\|_{H^{-1}(U; \mathbb{R}^m)}^2 dt \leq C \int_U |\mathbf{g}|^2 dx.$$

By the proof given in Section III.1 of [46], there is  $\mathbf{v}$  satisfying

$$\mathbf{v} \in L^\infty([0, T]; L^2(U; \mathbb{R}^m)) \cap L^2([0, T]; H_0^1(U; \mathbb{R}^m))$$

and subsequences  $(\mathbf{v}^{N_j})_{j \in \mathbb{N}}$  and  $(\mathbf{w}^{N_j})_{j \in \mathbb{N}}$  such that  $\mathbf{v}^{N_j} \rightarrow \mathbf{v}$  in  $L^2([0, T]; H_0^1(U; \mathbb{R}^m))$  and  $\partial_t(\mathbf{w}^{N_j}) \rightarrow \partial_t(D\psi(\mathbf{v}))$  in  $L^2([0, T]; H^{-1}(U; \mathbb{R}^m))$ . Moreover,  $\mathbf{v}$  satisfies the weak solution condition (A.7). Therefore, we are only left to verify that  $\mathbf{v}$  also satisfies (A.6).

To this end, we define

$$\mathbf{u}^N(\cdot, t) := \mathbf{v}^{k-1} + \left(\frac{t - \tau_k}{\tau}\right) (\mathbf{v}^k - \mathbf{v}^{k-1}), \quad \tau_{k-1} \leq t \leq \tau_k$$

for  $t \in [0, T]$ . By (A.12),

$$\sup_{0 \leq t \leq T} \int_U |D\mathbf{u}^N(x, t)|^2 dx + \int_0^T \int_U |\partial_t \mathbf{u}^N|^2 dx ds \leq C \int_U |D\mathbf{g}|^2 dx.$$

It follows that  $(\mathbf{u}^N(\cdot, t))_{N \in \mathbb{N}} \subset H_0^1(U; \mathbb{R}^m)$  is bounded for each  $t \in [0, T]$  and that  $\mathbf{u}^N : [0, T] \rightarrow L^2(U; \mathbb{R}^m)$  is uniformly equicontinuous. Since  $H_0^1(U; \mathbb{R}^m) \subset L^2(U; \mathbb{R}^m)$  with compact embedding, there we can pass to the limit along a subsequence  $(\mathbf{u}^{N_j})_{j \in \mathbb{N}}$  (that will not be labeled) to find a  $\mathbf{u}$  such that  $\mathbf{u}^{N_j} \rightarrow \mathbf{u}$  in  $C([0, T]; L^2(U; \mathbb{R}^m))$  (by Aubin's compactness theorem [3]) and  $\partial_t \mathbf{u}^{N_j} \rightharpoonup \partial_t \mathbf{u}$  in  $L^2(U \times (0, T); \mathbb{R}^m)$ . In particular, we note that  $\mathbf{u}$  satisfies (A.6).

We claim  $\mathbf{v} \equiv \mathbf{u}$ . To see this, we recall (A.12) and compute

$$\begin{aligned}
\int_0^T \int_U |\mathbf{u}^N - \mathbf{v}^N|^2 dx dt &= \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} \int_U |\mathbf{v}^N - \mathbf{u}^N|^2 dx dt \\
&= \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} \int_U \left| \left( 1 - \left( \frac{t - \tau_{k-1}}{\tau} \right) \right) (\mathbf{v}^k - \mathbf{v}^{k-1}) \right|^2 dx dt \\
&= \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} \left( 1 - \left( \frac{t - \tau_{k-1}}{\tau} \right) \right)^2 dt \int_U |\mathbf{v}^k - \mathbf{v}^{k-1}|^2 dx \\
&= \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} \left( \frac{\tau_k - t}{\tau} \right)^2 dt \int_U |\mathbf{v}^k - \mathbf{v}^{k-1}|^2 dx \\
&= \sum_{k=1}^N \frac{\tau}{3} \int_U |\mathbf{v}^k - \mathbf{v}^{k-1}|^2 dx \\
&= \frac{\tau^2}{3} \sum_{k=1}^N \int_U \frac{|\mathbf{v}^k - \mathbf{v}^{k-1}|^2}{\tau} dx \\
&\leq \frac{\tau^2}{3} \left( C \int_U |D\mathbf{g}|^2 dx \right) \\
&\leq \frac{T^2}{3N^2} \left( C \int_U |D\mathbf{g}|^2 dx \right).
\end{aligned}$$

Letting  $N = N_j$  and sending  $j \rightarrow \infty$  gives

$$\int_0^T \int_U |\mathbf{u} - \mathbf{v}|^2 dx dt = 0.$$

Consequently,  $\mathbf{v}$  satisfies (A.6) and is therefore a weak solution of (A.1).  $\square$

## References

- [1] Alt, H.; Luckhaus, S. *Quasilinear elliptic-parabolic differential equations*. Math. Z. 183 (1983), no. 3, 311–341.
- [2] Ambrosio, L.; Gigli, N.; Savaré, G. *Gradient flows in metric spaces and in the space of probability measures*. Second edition. Lectures in Mathematics ETH Zrich. Birkhuser Verlag, Basel, 2008.
- [3] Aubin, J.-P. *Un théorème de compacité*. C. R. Acad. Sci. Paris 256 (1963) 5042–5044.
- [4] Brézis, H. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Company (1973).

- [5] Campanato, S. *Proprietà di hölderianità di alcune classi di funzioni*. Ann. Scuola Norm. Sup. Pisa (3) 17 1963 175–188.
- [6] Campanato, S. *On the nonlinear parabolic systems in divergence form. Hölder continuity and partial Hölder continuity of the solutions*. Ann. Mat. Pura Appl. 137 (4) (1984) 83–122.
- [7] Chalmers, B. *Principles of Solidification*. Wiley, New York 1964.
- [8] Constantin, P.; Cordoba, D.; Gancedo, F.; Strain, R. *On the global existence for the Muskat problem*. J. Eur. Math. Soc. 15, 201–227.
- [9] Crandall, M. G.; Liggett, T. M. *Generation of semi-groups of nonlinear transformations on general Banach spaces*. Amer. J. Math. 93 1971 265–298.
- [10] Da Prato, G. *Spazi  $\mathcal{L}^{p,\theta}(\Omega, \delta)$  e loro proprietà*. Ann. Mat. Pura Appl. (4) 69 1965 383–392.
- [11] De Giorgi, E. *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*. Boll. Un. Mat. Ital. (4) 1 1968 135–137.
- [12] Di Nezza, E.; Palatucci, G.; Valdinoci, E. *Hitchhiker’s guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 (2012), no. 5, 521–573.
- [13] Diaz, J.; de Thélin, F. *On a nonlinear parabolic problem arising in some models related to turbulent flows*. SIAM J. Math. Anal. 25 (1994), no. 4, 1085–1111.
- [14] Duzaar, F.; Mingione, G. *Second order parabolic systems, optimal regularity, and singular sets of solutions*. Ann. I. H. Poincaré AN 22 (2005) 705–751.
- [15] Duzaar, F.; Mingione, G.; Steffen, K. *Parabolic systems with polynomial growth and regularity*. Mem. Amer. Math. Soc. 214 (2011), no. 1005.
- [16] Evans, L. C. *Partial differential equations*. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.
- [17] Evans, L. C.; Gariepy, R. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [18] Friedman, A. *The Stefan problem in several space variables*. Trans. Amer. Math. Soc. 133 1968 51–87.
- [19] Friedman, A. *Correction to: The Stefan problem in several space variables*. Trans. Amer. Math. Soc. 142 1969 557.
- [20] Giaquinta, M.; Martinazzi, L. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*. Second edition. Edizioni della Normale, Pisa (2012).

- [21] Giaquinta, M.; Struwe, M. *On the partial regularity of weak solutions of nonlinear parabolic systems*. Math. Z. 179 (1982), no. 4, 437–451.
- [22] Giusti, E.; Miranda, M. *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*. Boll. UMI 2 (1968), 1–8.
- [23] Gurtin, M. *On a theory of phase transitions with interfacial energy*. Arch. Rational Mech. Anal. 87 (1985), no. 3, 187–212.
- [24] Gurtin, M. *Multiphase thermo-mechanics with interfacial structure I. Heat conduction and the capillary balance law*. Arch. Rational Mech. Anal. 104 (1988) 195–221.
- [25] Hynd, R. *Partial regularity for type two doubly nonlinear parabolic systems*. In preparation.
- [26] Hynd, R. *Compactness methods for doubly nonlinear parabolic systems*. To appear, Transactions of the American Mathematical Society.
- [27] Imbert C.; Silvestre, L. *An introduction to fully nonlinear parabolic equations, An introduction to the Kähler-Ricci flow*. Lecture Notes in Math., vol. 2086, Springer, Cham, 2013, pp. 7–88.
- [28] Ivanov, A. *Regularity for doubly nonlinear parabolic equations*. J. Math. Sci. 83 (1) 1997 22–37.
- [29] Kuusi, T.; Laleoglu, R. Siljander, J.; Urbano, J. *Hölder continuity for Trudinger’s equation in measure spaces*. Calc. Var. Partial Differential Equations 45 (2012), no. 1-2, 193 – 229.
- [30] Lawson, H.B.; Osserman, R. *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*. Acta math. 139 (1977), 1–17.
- [31] Mingione, G. *The singular set of solutions to non-differentiable elliptic systems*. Arch. Ration. Mech. Anal. 166 (2003), no. 4, 287–301.
- [32] Oleĭnik, O. A. *A method of solution of the general Stefan problem*. Soviet Math. Dokl. 1 1960 1350–1354.
- [33] Perez, M. *Gibbs–Thomson effects in phase transformations*. Scripta Materialia 52 (2005) 709–712.
- [34] Porzio, M.; Vespi, V. *Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations*. J. Differential Equations 103 (1993), no. 1, 146–178.
- [35] Richardson, S. *Hele-Shaw flow with a free boundary produced by the injections of a fluid into a narrow channel*. J. Fluid. Mech. 56 (1972) 609–618.

- [36] Richardson, S. *Some Hele-Shaw flows with time-dependent free boundaries*. J. Fluid Mech. 102 (1981), 263–278.
- [37] Rogers, C. A. *Hausdorff Measures*. Cambridge University Press, Cambridge, 1970.
- [38] Rubinstein, L. *On the determination of the position of the boundary which separates two phases in the one-dimensional problem of Stefan*. Dokl. Acad. Nauk USSR 58 (1947) 217–220.
- [39] Saffman, P. G.; Taylor, G. *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*. Proc. Roy. Soc. London. Ser. A 245 1958 312–329.
- [40] Simon, J. *Compact sets in the space  $L^p(0, T; B)$* . Ann. Mat. Pura Appl. (4) 146 (1987), 65–96.
- [41] Stefan, J. *Über einige Probleme der Theorie der Wärmeleitung*. Sitzungber., Wien, Akad. Mat. Natur. 98 (1889) 473–484.
- [42] Temam, R. *Navier-Stokes equations. Theory and numerical analysis*. Revised edition. Studies in Mathematics and its Applications, 2. North-Holland Publishing Co., Amsterdam-New York, 1979.
- [43] Trokhimtchouk, M. *Everywhere regularity of certain nonlinear diffusion systems*. Calc. Var. Partial Differential Equations 37 (2010), no. 3-4, 407–422.
- [44] Trudinger, N. *Pointwise estimates and quasilinear parabolic equations*. Comm. Pure Appl. Math. 21 1968 205–226.
- [45] Vespri, V. *On the local behaviour of solutions of a certain class of doubly nonlinear parabolic equations*. Manuscripta Math. 75 (1992), no. 1, 65–80.
- [46] Visintin, A. *Models of phase transitions*. Progress in Nonlinear Differential Equations and their Applications, 28. Birkhäuser Boston, Inc., Boston, MA, 1996.
- [47] Woodruff, P. *The Solid-Liquid Interface*. Cambridge University Press, Cambridge 1973.